MEDDELANDEN

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LUNDS ASTRONOMISKA OBSERVATORIUM

SERIE II

N:r 25

ON THE DISTANCES AND LUMINOSITIES OF STARS OF SPECTRAL TYPE G AS DERIVED FROM THEIR PROPER MOTIONS

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LIC. PHIL.

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INTRODUCTION.

Characteristic of stars of the spectral type G is the frequent occurrence of very large proper motions, a fact by which a statistic investigation on the motions of these stars is rendered more difficult, considering that a great number of stars must be rejected, which from a statistical point of view is inappropriate as by this means systematical errors cannot be avoided.

The method, applied in the first chapter for determining the parameter R presumes a somewhat constant absolute magnitude. This, however, cannot be assumed to be the case with the G stars. The result achieved in the first chapter shows that R is very sensible of the rejection of stars with great proper motions. Yet, from the value of R obtained I have computed the velocity ellipsoid for the apparently brighter stars. The form and the position of the ellipsoid are in fairly good accordance with those found in similar investigations of the other spectral types made at this observatory.

As the G stars exhibit such great anomalies they are better treated individually. This has been made possible here thanks to a method of computing from the proper motions of a star the most probable value of its distance. This method was given by prof. CHARLIER in his lectures during the spring term of 1917, and with his kind permission I publish it here.

In the second chapter I have expounded this method and further developed it for numerical applications on the supposition that the velocity ellipsoid is an ellipsoid of rotation.

The method presumes that the frequency distribution $\Phi(s)$ of the velocity components perpendicular to the line of sight can be computed for all stars moving in a certain direction. In determining the characteristics of this $\Phi(s)$ I have followed the indications given by prof. CHARLIER in his above-mentioned lectures. In section C of the same chapter I have shown how the velocity ellipsoid may be computed specially for this purpose.

In the last chapter section A, the method has been applied to stars $< 6^{m}.0$ and in section B to the parallax stars of ADAMS and JOY's catalogue. Here there appears a certain systematical difference between the determination of the absolute magnitudes from the probable distance and the determination made by ADAMS and JOY, a difference which, however, finds an explanation in the well-known fact that the velocity of a star is correlated with its absolute magnitude. In the next section the method has been applied to the stars of the Greenwich catalogue, and the result shows that, when we have to deal with apparently faintes stars, the dispersion of the absolute magnitude is smaller. I have therefore applied to these the same method as in chapter I.

The units for length and time used here are siriometer and stellar year introduced by CHARLIER:—

1 siriometer (sir.) = 10^6 times the mean distance from the earth to the sun,
1 stellar year (st.) = 10^6 tropical years.

The absolute magnitude is defined in this investigation according to CHARLIER as the apparent magnitude of a star at a distance of one siriometer.

Another unit for length used in several papers is the parsec., which corresponds to a parallax of one second of arc.:—

1 parsec. = 0.206 sir.

The apparent magnitude at a distance of 1 sir. = the apparent magnitude at 10 parsecs. $-1^m.57$.

The following abbreviations are used:—

A. J. = The Astronomical Journal,
Ap. J. = The Astrophysical Journal,
G. P. = Publications of the Astronomical Laboratory at Groningen,
L. M. I. = Meddelanden från Lunds Observatorium, Serie I,
L. M. II. = « « « « « II,
M. N. = Monthly Notices of the Royal Astronomical Society.

CHAPTER I.

THE PARAMETER R AND THE VELOCITY DISTRIBUTION.

1. The method, here applied, is developed by Prof. CHARLIER in "Meddelande från Lunds astronomiska observatorium", No. 14, Ser. 11, and is moreover used in several investigations. Prof. CHARLIER has applied the method to the stars of spectral type *B*. Later the same method has also been applied to the stars of other spectral types, the stars of type *O* by GYLLENBERG, the stars of type *A* and *F* by MALMQUIST and LUNDAHL respectively.[1]) I will here give only a short description of the theory and a recapitulation of the formulae.

Let m be the apparent magnitude of a star, then the distance r may be written

$$r = R \cdot 10^{0.2\,m}. \tag{1}$$

Here R denotes the distance at which the star should have the apparent magnitude 0. This parameter mainly depends on the temperature and the diameter of the star. As the spectral type is closely correlated with the temperature, the latter may be considered as constant for a certain spectral type. Further, the diameter may be supposed to be a constant when the stars are of that type where the temperature is the highest, according to the theory that only the big stars are able to reach the highest temperature.

Thus if we suppose R to be a constant for each spectral class, this assumption very well accords with the actual fact if the stars are of the type *B*, but the accordance gets worse when we pass through the series from the earliest to the latest types in the order:—*BAFGKM* because of the phenomenon of dwarf and giant stars.

If there is any way to determine the constant R, then the distance of each star may be found from the formula (1).

The computation of R is, according to CHARLIER, performed in the following way.

If we assume UVW to be the components of velocity of a star in a system of right angular coordinates, having its axis of Z directed along the radius vector to the star, the positive X-axis in the direction of increasing right ascension, and the positive Y-axis in the direction of increasing declination, if, further, we assume $U''V''W''$ to be the components in the usual astronomical system and let γ_{ij} be the direction cosines between the two systems, we have

$$\begin{aligned} U &= \gamma_{11}U'' + \gamma_{21}V'' + \gamma_{31}W'', \\ V &= \gamma_{12}U'' + \gamma_{22}V'' + \gamma_{32}W'', \\ W &= \gamma_{13}U'' + \gamma_{23}V'' + \gamma_{33}W'', \end{aligned} \tag{2}$$

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[1]) L. M. I, 75, 76 and 77. L. M. II, 21 and 22.

where γ_{ij} are functions of α und δ only

$$\begin{aligned} \gamma_{11} &= -\sin\alpha, & \gamma_{21} &= +\cos\alpha, & \gamma_{31} &= 0, \\ \gamma_{12} &= -\sin\delta\cos\alpha, & \gamma_{22} &= -\sin\delta\sin\alpha, & \gamma_{32} &= +\cos\delta, \\ \gamma_{13} &= +\cos\delta\cos\alpha, & \gamma_{23} &= +\cos\delta\sin\alpha, & \gamma_{33} &= +\sin\delta. \end{aligned}$$

Let $u = \Delta\alpha\cos\delta$ and $v = \Delta\delta$ be the proper motions of the star in right ascension and declination respectively. We then have

$$U = Ru' \text{ and } V = Rv',$$

where

$$u' = 10^{0.2\,m}\cdot u \text{ and } v' = 10^{0.2\,m}\cdot v.^{1})$$

W is the radial velocity. Then the quantities u', v' and W are known for each star. Having a number of stars the proper motion and radial velocity of which are known, we get from the two first equations (2) the mean values of $\frac{U''}{R}$, $\frac{V''}{R}$, $\frac{W''}{R}$ and from the third the mean values of $U''V''W''$ by the method of least squares. When we combine these two results, we get the mean value of $\frac{1}{R}$. The inverse value of $M\left(\frac{1}{R}\right)$ may be approximately substituted for $M(R)$.

The mean values of $U''V''W''$ are the mean values of the velocities of the stars referred to the sun. Suppose that the velocity of the sun is known, then the third equation in (2) is unnecessary for the determination of R. When Ω_0 is the velocity of the sun, we have

$$\frac{\Omega_0}{R} = \sqrt{\left(\frac{U''}{R}\right)^2 + \left(\frac{V''}{R}\right)^2 + \left(\frac{W''}{R}\right)^2}. \tag{3}$$

R being known, we get U and V from the relations

$$U = Ru' \text{ and } V = Rv',$$

and by the direction cosines γ_{ij} we get the components $U''V''W''$ for each star.

2. The material treated here is stars of type G, which are brighter than $6^{m}_{\cdot}0$. The proper motion is obtained from BOSS' catalogue, and the magnitude and spectral type from Harvard 50. These stars were first divided into subclasses and into stars brighter than $5^{m}_{\cdot}0$ and fainter than $5^{m}_{\cdot}0$.

I have found several stars with very large proper motion. It was necessary to reject some of these stars, because their influence on the value of R is very great. In this rejection I have proceeded in the following manner. I have classified the different values of u' and v' respectively according to a certain class breadth (1″). If three vacant classes succeeded each other, I rejected those stars for which the values

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[1]) If u and v are given in seconds of arc per year, we have to multiply by $\frac{10^6}{206265} = 4.8481$ to get them in radians per stellar year.

of u' and v' respectively lay outside. The number of rejected stars is shown in the third column of table I.

TABLE I.

	N	*Rej.*	$\frac{\Omega_0}{R}$	$\varepsilon\left(\frac{\Omega_0}{R}\right)$	R	A	D	M
G_0 5.00–5.99	126	9	3.045	1.154	1.80	283°.83	+ 57°.79	— 0.570
$G_0 \leq 4.99$	59	6	3.803	1.290	1.04	292°.02	+ 17°.16	— 0.085
G_2 5.00–5.99	32	2	3.129	1.287	1.27	267°.26	+ 31°.88	— 0.519
$G_2 \leq 4.99$	3	3						
G_5 5.00–5.99	125	5	3.879	0.741	1.01	270°.24	+ 27°.80	— 0.022
$G_5 \leq 4.99$	71	8	1.315	0.504	3.01	218°.68	+ 42°.80	— 2.393
G_8 5.00–5.99	1	1						
G_0 5.00–5.99	126	17	1.777	0.834	2.23	278°.89	+ 62°.82	— 1.742
$G_0 \leq 4.99$	59	11	1.534	0.795	2.58	274°.84	+ 43°.92	— 2.058

In the fourth column I give the value $\frac{\Omega_0}{R}$, obtained in using the two first equations in (2). The next column shows the mean errors of $\frac{\Omega_0}{R}$. The quantity R in the following column is obtained from the value 3.96 sir./st. as the velocity of the sun. This value (3.96 ± 0.474) was found by Dr. GYLLENBERG in "Meddelande", Ser. II, No. 13, from 208 G-stars with known radial velocity. The 9th column contains the absolute magnitude

$$M = -5 \log R.$$

We find that the value R for the group $G_5 \leq 4.99$ differs to an essential degree from the other groups, where the values of R are more comparable.

The method of rejection was, however, not effective. I therefore applied a new method. I computed the dispersion of u' and v' respectively, and rejected those stars for which the value u' and v' respectively lay outside three times the dispersion. In this manner the number of stars in the group $G_0 \leq 4.99$ was reduced by 5 stars and the group G_0 5.00–5.99 by 8 stars. The number of stars in the other groups remains unchanged. The new values of R are shown below in table I.

We find that the influence of the rejection is very important. According to RUSSEL and HERTZSPRUNG we may suppose that there are giant and dwarf stars. The dwarf stars generally have larger proper motion than the giant stars, and then the rejected stars probably contain more of the former. It is therefore to be expected that the value of R will increase. The method of determining R primarily gives the mean value of $\frac{1}{R}$. When the values of R for the rejected stars are small, its influence on the mean value is very great.

There are about 30 stars in Groningen catalogue[1]) of the spectral type G with known parallax. The main part of these (21 stars) are among the rejected stars. For each of these the parameter R is obtained from the formula

$$R = \frac{0.206265}{\pi} \cdot 10^{-0.2m}.$$

[1]) G. P. 24.

TABLE II.

	Gr. No.	Magn.	π	R
$G_0 \leq 4.99$	7	2.90	0.143	0.379
	41	4.17	0.112	0.270
	68	4.85	0.111	0.199
	156	4.41	0.179	0.151
	173	4.32	0.102	0.909
	181	4.32	0.116	0.801
	199	0.33	0.759	0.233
G_0 5.00–5.99	34	5.07	0.119	0.168
	64	5.04	0.016	1.263
	211	5.18	0.094	0.201
	250	5.36	0.123	0.156
	302	5.89	0.104	0.131
	359	5.85	0.067	0.209
$G_5 \leq 4.99$	44	4.30	0.162	0.175
	45	4.40	0.069	0.395
	50	4.48	0.174	0.151
	260	3.48	0.106	0.392
G_5 5.00–5.99	17	5.26	0.112	0.164
	29	5.74	0.061	0.240
	36	5.92	0.143	0.094
	287	5.23	0.065	0.285

The values of R are shown in table II. They are very small. Taking the mean value, and considering the weight as proportional to the quantity π^4, we get

$$R = 0.22 \pm 0.172$$

and

$$M = +3.29.$$

When the mean was taken, the star No. 199 was excluded on account of too great weight, but the value of R for this star is almost identical with the computed mean.

From the quantities $\frac{U''}{R}$, $\frac{V''}{R}$, $\frac{W''}{R}$ the position of the apex is obtained by the relations

$$\frac{U''}{R} = \cos A \cos D,$$

$$\frac{V''}{R} = \sin A \cos D,$$

$$\frac{W''}{R} = \sin D.$$

The 7th and 8th columns in table I give the value of A and D.

3. *The velocity distribution.* In examining the stars with known radial velocity obtained from a catalogue made by GYLLENBERG, I found the main part of the stars brighter than 5^m_0. I have therefore computed the velocity distribution of the groups $G_0 \leq 4.99$ and $G_5 \leq 4.99$, and in the computation I have used a common value of R, obtained by the addition of the normal equations of the stars $G_0 \leq 4.99$ ($N = 48$) and $G_5 \leq 4.99$ ($N = 63$). The value thus obtained is

$$R = 2.984 \text{ sir./st.}$$

When R is known, we get the velocity components U and V from the relations $U = Ru'$; $V = Rv'$, and from the components U, V, W we deduce the components U'', V'', W'' by use of the direction cosines γ_{ij}. From CHARLIER's investigation we know the position of the galactic system. Using his value of the direction cosines, e_{ij}, which connects this system with the equatorial system, we get the velocity components relatively to the sun $U^\circ V^\circ W^\circ$ in the galactic system.

When the galactic components are known the velocity ellipsoid is derived from the moments in the following manner.

First we form the absolute moments about the origin

$$\mu'_i = \Sigma x_i, \quad \mu'_{ij} = \Sigma x_i x_j \qquad (x_1 = U^\circ, \ x_2 = V^\circ, \ x_3 = W^\circ).$$

Then the relative moments about the origin.

$$\nu'_i = \frac{\Sigma x_i}{N} = m_i,$$

$$\nu'_{ij} = \frac{\Sigma x_i x_j}{N},$$

where N is the number of stars. Then the relative moments about the mean

$$\nu_{ij} = \nu'_{ij} - m_i m_j.$$

From ν_{ij} we get the dispersion and correlation coefficients

$$\sigma_i = \sqrt{\nu_{ij}},$$

$$r_{ij} = \frac{\nu_{ij}}{\sigma_i \sigma_j}.$$

Let S be the determinant

$$S = |r_{ij}| \qquad (r_{ii} = 1)$$

and

$$S_{ij} = \frac{\partial S}{\partial r_{ij}}.$$

Then we form the quantities a_{ij} by the relations

$$a_{ij} = \frac{S_{ij}}{S \cdot \sigma_i \cdot \sigma_j}.$$

Now all the quantities necessary for the forming of the frequency function are known. We get

$$F = \frac{1}{\sigma_1 \sigma_2 \sigma_3 \sqrt{(2\pi)^3 S}} \cdot e^{-\frac{1}{2}\Sigma a_{ij} x_i x_j}.$$

The position and magnitude of the principal axes we get from the Jacobian determinant

$$I(\lambda) = \begin{vmatrix} a_{11} - \lambda, & a_{12}, & a_{13} \\ a_{21}, & a_{22} - \lambda, & a_{23} \\ a_{31}, & a_{32}, & a_{33} - \lambda \end{vmatrix}$$

and the equations:—

$$\begin{aligned} (a_{11} - \lambda_i)\, l + a_{12}\, m + a_{13}\, n &= 0, \\ a_{21}\, l + (a_{22} - \lambda_i)\, m + a_{23}\, n &= 0, \\ a_{31}\, l + a_{32}\, m + (a_{33} - \lambda_i)\, n &= 0 \qquad (i = 1, 2, 3). \end{aligned}$$

λ_i are the roots of the equation $I(\lambda) = 0$ and l, m, n the galactic direction cosines of the principal axes. The magnitude of the axes α_1, α_2, α_3 we have from the relations

$$\alpha_i = \frac{1}{\sqrt{\lambda_i}}.$$

On examining the material of stars with known radial velocity I found it necessary to reject a few stars with large velocity components. After computing the equatorial components, I rejected those stars where one of the components U'' V'' W'' exceeded 20 sir./st. There remain 100 stars, 39 of the group $G_0 \leq 4^m.99$ and 61 of the group $G_5 \leq 4^m.99$. The results obtained are:—

$$\nu_1' = m_1 = -2.418,$$
$$\nu_2' = m_2 = +1.411,$$
$$\nu_3' = m_3 = -1.766.$$

The solar motion Ω_0, given from the relation $\Omega_0 = \sqrt{m_1^2 + m_2^2 + m_3^2}$, is

$$\Omega_0 = 3.81.$$

The dispersion and correlation coefficients are:—

$$\sigma_1 = 4.14 \pm 0.293, \qquad r_{12} = -0.2409,$$
$$\sigma_2 = 5.71 \pm 0.404, \qquad r_{13} = +0.2328,$$
$$\sigma_3 = 3.06 \pm 0.217. \qquad r_{23} = +0.0596.$$

The roots of the Jacobian equation are:—

$$\lambda_1 = +0.1247,$$
$$\lambda_2 = +0.0291,$$
$$\lambda_3 = +0.0601.$$

The following table gives the position and magnitude of the axes:—

	RA	Decl.	a_i
λ_1	358°.71	− 5°.89	2.83
λ_2	267°.30	−12°.80	5.87
λ_3	113°.51	−75°.96	4.08

The velocity ellipsoid is not a rotation ellipsoid. The greatest axis corresponds to the vertex. The shortest axis is perpendicular to the galactic plane. MALMQUIST[1]) and LUNDAHL[2]) have got the same result, while GYLLENBERG in his investigation on the stars with known radial velocity[3]) has obtained a contrary result. He got the shortest axis in the galactic plane.

4. By way of experiment I computed the velocity ellipsoid after rejecting the stars whose equatorial components were greater than 3 times the dispersion. In this way another 5 stars where rejected. I now got the following results:—

$$\nu_1' = m_1 = -1.992,$$
$$\nu_2' = m_2 = +1.016,$$
$$\nu_3' = m_3 = -1.519.$$

[1]) L. M. I, 76.
[2]) L. M. I, 77.
[3]) L. M. II, 13.

The velocity of the sun is now

$$\Omega_0 = 2.70.$$

The dispersion and correlation coefficients are:—

$$\sigma_1 = 3.70 \pm 0.268, \qquad r_{12} = -0.2811,$$
$$\sigma_2 = 4.64 \pm 0.337, \qquad r_{13} = -0.0114,$$
$$\sigma_3 = 2.89 \pm 0.178. \qquad r_{23} = +0.0801.$$

The roots of the Jacobian equation:—

$$\lambda_1 = +0.1771,$$
$$\lambda_2 = +0.0480,$$
$$\lambda_3 = +0.0831.$$

The direction and magnitude of the principal axes:—

	RA	Decl.	α_i
λ_1	7°15	−26°36	2.38
λ_2	271°98	−10°21	4.82
λ_3	157°44	−60°68	3.47

We find that the form and the position of the ellipsoid is the same, but its coincidence with the galactic system is better. The angles formed by the axes of the velocity ellipsoid and the *corresponding* axes of the galactic system before and after rejecting the 5 stars are:—

N	*X*	*Y*	*Z*
100	29°21	17°75	23°47
95	19°98	22°70	3°58

On the other hand the velocity of the sun is now very small. This is obviously due to the fact that the quantity R has been computed from more stars than I used when computing the velocity ellipsoid, and that the stars rejected are stars of large proper motions.

Though the *G*-stars show a large dispersion in the absolute luminosity, the results of the determination of the velocity distribution are in good accordance with the preceding investigations of the other types. The question of the distribution of the absolute luminosity is, however, the most interesting. To get an idea of this I have computed the probable distance of every star of the spectral type *G* with known proper motion and brighter than the magnitude 6.0 by using a method, given by Prof. CHARLIER during his lectures in the spring term of 1917, which method I will expound in the following chapter.

CHAPTER II.

A. THE DETERMINATION OF THE PROBABLE DISTANCE OF A STAR FROM ITS PROPER MOTION.

5. In the following study (except the last chapter section C) I have used the galactic systems of coordinates instead of the equatorial system. By this means we obtain a better survey of the connection between the results achieved and the positions of the stars in relation to the galactic plane.

In dealing with a large material it is almost necessary to simplify calculations by dividing the heavens into different squares and assuming the stars to be situated in the centres. Such a division of the sky has been proposed by Prof. CHARLIER[1]); it was first made with the equatorial systems as a basis, but later a division based on the galactic coordinates has been made. A description of these squares has been given by MALMQUIST.[2])

I use the galactic system defined by PICKERING. The direction cosines relating the axes of this system (K_g) with those of the common equator system (K'') are:—

		K'': X''	Y''	Z''
	X_g	ε_{11}	ε_{21}	ε_{31}
K_g:	Y_g	ε_{12}	ε_{22}	ε_{32}
	Z_g	ε_{13}	ε_{23}	ε_{33}

$\varepsilon_{11} = +0.1736$, $\varepsilon_{21} = -0.9848$, $\varepsilon_{31} = 0$,
$\varepsilon_{12} = +0.4623$, $\varepsilon_{22} = +0.0815$, $\varepsilon_{32} = +0.8829$,
$\varepsilon_{13} = -0.8695$, $\varepsilon_{23} = -0.1533$, $\varepsilon_{33} = +0.4695$.

Besides this system of coordinates we make use of the following system:—

K_1 having its axes coincident with those of the velocity ellipsoid and

K_0 having its axis of Z directed along the radius vector to the centre of gravity of a square its axis of X in the direction of increasing galactic longitude and its axis of Y in the direction of increasing galactic latitude.

The direction cosines of the systems K_1 and K_0 referred to the system K_g are:—

		K_g: X_g	Y_g	Z_g
	X_0	β_{11}	β_{21}	β_{31}
K_0:	Y_0	β_{12}	β_{22}	β_{32}
	Z_0	β_{13}	β_{23}	β_{33}

		K_g: X_g	Y_g	Z_g
	X_1	α_{11}	α_{21}	α_{31}
K_1:	Y_1	α_{12}	α_{22}	α_{32}
	Z_1	α_{13}	α_{23}	α_{33}

[1]) L. M. II, 8.
[2]) L. M. II, 22.

Here β_{ij} have the same numerical values as the corresponding γ_{ij} for the equatorial squares and these quantities are to be found in L. M. II, 12 and 13.

α_{ij} are known when the velocity ellipsoid is known.

$$\begin{aligned}
\alpha_{11} &= \cos B_1 \cos L_1, & \alpha_{21} &= \cos B_1 \sin L_1, & \alpha_{31} &= \sin B_1,\\
\alpha_{12} &= \cos B_2 \cos L_2, & \alpha_{22} &= \cos B_2 \sin L_2, & \alpha_{32} &= \sin B_2,\\
\alpha_{13} &= \cos B_3 \cos L_3, & \alpha_{23} &= \cos B_3 \sin L_3, & \alpha_{33} &= \sin B_3,
\end{aligned}$$

where $L_1 B_1 L_2 B_2$ and $L_3 B_3$ are the galactic longitude and latitude of the direction of the three axes.

In the system K_1 the velocity ellipsoid is devoid of the rectangle terms. Thus if $U'V'W'$ are the velocity components of a star in this system, the velocity-distribution may be written

$$F(U',V',W') = H \cdot e^{-\frac{1}{2}f},$$

where

$$f = A'(U'-U'_0)^2 + B'(V'-V'_0)^2 + C'(W'-W'_0)^2$$

and H = const.

The constants A', B' and C' are formed from the dispersions through the relations

$$A' = 1:\sigma_1^2, \quad B' = 1:\sigma_2^2, \quad C' = 1:\sigma_3^2.$$

$U'_0 V'_0 W'_0$ are the velocity components of the "stream". The stream velocity is equal to the solar velocity with conversed sign.

In the system K_g the ellipsoid takes the form

$$\begin{aligned}
f = {} & A_g(U_g-U_g^\circ)^2 + B_g(V_g-V_g^\circ)^2 + C_g(W_g-W_g^\circ)^2 + \\
& + 2D_g(V_g-V_g^\circ)(W_g-W_g^\circ) + 2E_g(U_g-U_g^\circ)(W_g-W_g^\circ) + 2F_g(U_g-U_g^\circ)(V_g-V_g^\circ).
\end{aligned}$$

$U_g V_g W_g$ are the components in the system K_g. The coefficients $A_g B_g C_g D_g E_g$ and F_g may be performed by means of the direction cosines α_{ij}.

$$(1)\quad \left\{\begin{aligned}
A_g &= A'\alpha_{11}^2 + B'\alpha_{12}^2 + C'\alpha_{13}^2,\\
B_g &= A'\alpha_{21}^2 + B'\alpha_{22}^2 + C'\alpha_{23}^2,\\
C_g &= A'\alpha_{31}^2 + B'\alpha_{32}^2 + C'\alpha_{33}^2,\\
D_g &= A'\alpha_{21}\alpha_{31} + B'\alpha_{22}\alpha_{32} + C'\alpha_{23}\alpha_{33},\\
E_g &= A'\alpha_{11}\alpha_{31} + B'\alpha_{12}\alpha_{32} + C'\alpha_{13}\alpha_{33},\\
F_g &= A'\alpha_{11}\alpha_{21} + B'\alpha_{12}\alpha_{22} + C'\alpha_{13}\alpha_{23}.
\end{aligned}\right.$$

In the system K_0 the ellipsoid takes a similar form.

$$\begin{aligned}
(2)\qquad f = {} & A(U-U_0)^2 + B(V-V_0)^2 + C(W-W_0)^2 + \\
& + 2D(V-V_0)(W-W_0) + 2E(U-U_0)(W-W_0) + 2F(U-U_0)(V-V_0),
\end{aligned}$$

where

$$(3)\quad \left\{\begin{aligned}
A &= A_g\beta_{11}^2 + B_g\beta_{21}^2 + C_g\beta_{31}^2 + 2D_g\beta_{21}\beta_{31} + 2E_g\beta_{31}\beta_{11} + 2F_g\beta_{11}\beta_{21},\\
B &= A_g\beta_{12}^2 + B_g\beta_{22}^2 + C_g\beta_{32}^2 + 2D_g\beta_{22}\beta_{32} + 2E_g\beta_{32}\beta_{12} + 2F_g\beta_{12}\beta_{22},\\
C &= A_g\beta_{13}^2 + B_g\beta_{23}^2 + C_g\beta_{33}^2 + 2D_g\beta_{23}\beta_{33} + 2E_g\beta_{33}\beta_{13} + 2F_g\beta_{13}\beta_{23},\\
D &= A_g\beta_{12}\beta_{13} + B_g\beta_{22}\beta_{23} + C_g\beta_{32}\beta_{33} + D_g(\beta_{22}\beta_{33} + \beta_{32}\beta_{23}) + E_g(\beta_{32}\beta_{13} + \beta_{33}\beta_{12}) + F_g(\beta_{12}\beta_{23} + \beta_{13}\beta_{22}),\\
E &= A_g\beta_{13}\beta_{11} + B_g\beta_{23}\beta_{21} + C_g\beta_{33}\beta_{31} + D_g(\beta_{23}\beta_{31} + \beta_{33}\beta_{21}) + E_g(\beta_{33}\beta_{11} + \beta_{31}\beta_{13}) + F_g(\beta_{13}\beta_{21} + \beta_{13}\beta_{23}),\\
F &= A_g\beta_{11}\beta_{12} + B_g\beta_{21}\beta_{22} + C_g\beta_{31}\beta_{32} + D_g(\beta_{21}\beta_{32} + \beta_{31}\beta_{22}) + E_g(\beta_{31}\beta_{12} + \beta_{32}\beta_{11}) + F_g(\beta_{11}\beta_{22} + \beta_{12}\beta_{21}).
\end{aligned}\right.$$

The frequency function is now of the form

$$F = H \cdot e^{-\frac{1}{2}f}.$$

If only the proper motion, and not the radial velocity, is taken into consideration, we have to integrate the function F over W between $-\infty$ and $+\infty$. Thus we get

$$\int_{-\infty}^{+\infty} F\, dW = H_1 \cdot e^{-\frac{1}{2}f_1}.$$

The function f_1 is of the form[1])

$$f_1 = A_1 (U - U_0)^2 + B_1 (V - V_0)^2 + 2 F_1 (U - U_0)(V - V_0).$$

The equations for determining the coefficients A_1, B_1 and F_1 are:—

$$\begin{aligned}
A_1 C = {} & B'C' (\alpha_{11}\beta_{12} + \alpha_{21}\beta_{22} + \alpha_{31}\beta_{32})^2 \\
& + C'A' (\alpha_{12}\beta_{12} + \alpha_{22}\beta_{22} + \alpha_{32}\beta_{32})^2 \\
& + A'B' (\alpha_{13}\beta_{12} + \alpha_{23}\beta_{22} + \alpha_{33}\beta_{32})^2, \\
B_1 C = {} & B'C' (\alpha_{11}\beta_{11} + \alpha_{21}\beta_{21} + \alpha_{31}\beta_{31})^2 \\
& + C'A' (\alpha_{12}\beta_{11} + \alpha_{22}\beta_{21} + \alpha_{32}\beta_{31})^2 \\
& + A'B' (\alpha_{13}\beta_{11} + \alpha_{23}\beta_{21} + \alpha_{33}\beta_{31})^2, \\
F_1 C = {} & - B'C' (\alpha_{11}\beta_{11} + \alpha_{21}\beta_{21} + \alpha_{31}\beta_{31})(\alpha_{11}\beta_{12} + \alpha_{21}\beta_{22} + \alpha_{31}\beta_{32}) \\
& - C'A' (\alpha_{12}\beta_{11} + \alpha_{22}\beta_{21} + \alpha_{32}\beta_{31})(\alpha_{12}\beta_{12} + \alpha_{22}\beta_{22} + \alpha_{32}\beta_{32}) \\
& - A'C' (\alpha_{13}\beta_{11} + \alpha_{23}\beta_{21} + \alpha_{33}\alpha_{31})(\alpha_{13}\beta_{12} + \alpha_{23}\beta_{22} + \alpha_{33}\beta_{32}),
\end{aligned}$$

where

$$\begin{aligned}
C = {} & A' (\alpha_{11}\beta_{13} + \alpha_{21}\beta_{23} + \alpha_{31}\beta_{33})^2 \\
& + B' (\alpha_{12}\beta_{13} + \alpha_{22}\beta_{23} + \alpha_{32}\beta_{33})^2 \\
& + C' (\alpha_{13}\beta_{13} + \alpha_{23}\beta_{23} + \alpha_{33}\beta_{33})^2.
\end{aligned}$$

If we introduce polarcoordinates by the relations

$$U = s \cos\theta,$$
$$V = s \sin\theta,$$

the frequency function

$$H_1\, dU\, dV \cdot e^{-\frac{1}{2}f_1}$$

takes the form

$$H_1 \cdot s\, ds\, d\theta\, e^{-\frac{1}{2}f_2},$$

where

$$f_2 = A_1 (s\cos\theta - U_0)^2 + B_1 (s\sin\theta - V_0)^2 + 2 F_1 (s\cos\theta - U^0) \cdot (s\sin\theta - V_0).$$

The function f_2 may be written

$$-\frac{1}{2} f_2 = -\frac{1}{2} a s_2 + b s - \frac{1}{2} c,$$

[1]) Compare CHARLIER: "Studies in Stellar Statistics", L. M. II, 9, page 82.

where

$$\begin{aligned} a &= A_1 \cos^2\theta + B_1 \sin^2\theta + 2 F_1 \cos\theta \sin\theta, \\ b &= A_1 U_0 \cos\theta + B_1 V_0 \sin\theta + F(\cos\theta V_0 + \sin\theta U_0), \\ c &= A_1 U_0^2 + B_1 V_0^2 + 2 F_1 U_0 V_0. \end{aligned}$$

Here U_0 and V_0 are components of the stream velocity and these quantities can be computed, if the solar motion and the direction of Apex are known. Further A_1 B_1 and F_1 are to be determined from the velocity ellipsoid.

If we take into consideration a certain square and a certain direction $\theta \pm \frac{1}{2} d\theta$, then the number of stars, which move in this direction and whose linear cross motions lie between $s \pm \frac{1}{2} ds$, is proportional

$$ds \cdot s e^{-\frac{1}{2} a s^2 + b s - \frac{1}{2} c} = ds\, \Phi(s).$$

If the proper motion $\Delta b \operatorname{Cos} b$ and Δb of a star is known, the angle θ may be computed from

$$\begin{aligned} \Delta l \cos b &= \mu \cos\theta, \\ \Delta l &= \mu \sin\theta. \end{aligned}$$

Then the parameters of the function Φ are known. The value of s which makes Φ a maximum, may be regarded as the probable value of the linear proper motion of the star. This value may be denoted s_0 and is found by taking the logarithm of Φ and then deriving

$$0 = \frac{1}{s_0} - a s_0 + b. \tag{4}$$

At last the probable distance r of the star is found by putting

$$s_0 = \mu \cdot r,$$

where

$$\begin{aligned} \mu &= \sqrt{u^2 + v^2}, \\ u &= \Delta l \cos b,\ v = \Delta b. \end{aligned}$$

If u and v are expressed in radians per stellar year, we get the distance r expressed in siriometers.

6. *The velocity distribution is an ellipsoid of revolution.* If two of the axes of the velocity ellipsoid have the same value, say $A' = B'$, the equations for determining the coefficients A_1 B_1 and F_1 take the form:—

$$\begin{aligned} A_1 C &= B' [C' + (B' - C')(\alpha_{13}\beta_{12} + \alpha_{23}\beta_{22} + \alpha_{33}\beta_{32})^2], \\ B_1 C &= B' [C' + (B' - C')(\alpha_{13}\beta_{11} + \alpha_{23}\beta_{21} + \alpha_{33}\beta_{31})^2], \\ F_1 C &= -B'(B' - C')(\alpha_{13}\beta_{12} + \alpha_{23}\beta_{22} + \alpha_{33}\beta_{32})(\alpha_{13}\beta_{11} + \alpha_{23}\beta_{22} + \alpha_{33}\beta_{32}), \\ C &= B' - (B' - C')(\alpha_{13}\beta_{13} + \alpha_{23}\beta_{23} + \alpha_{33}\beta_{33}). \end{aligned}$$

Suppose the longest semiaxis to be λ and the other two $\alpha\lambda$, we have

$$C' = \frac{1}{\lambda^2} \text{ and } B' = \frac{1}{\alpha^2\lambda^2}.$$

Further we put

$$\begin{aligned} \gamma_1 &= \alpha_{13}\beta_{11} + \alpha_{23}\beta_{21} + \alpha_{33}\beta_{31}, \\ \gamma_2 &= \alpha_{13}\beta_{12} + \alpha_{23}\beta_{22} + \alpha_{33}\beta_{32}, \\ \gamma_3 &= \alpha_{13}\beta_{13} + \alpha_{23}\beta_{23} + \alpha_{33}\beta_{33}. \end{aligned} \tag{5}$$

We thus get

$$\lambda^2 A_1 = \frac{\alpha^2 + (1-\alpha^2)\gamma_2^2}{\alpha^2[1-(1-\alpha^2)\gamma_3^2]},$$

$$\lambda^2 B_1 = \frac{\alpha^2 + (1-\alpha^2)\gamma_1^2}{\alpha^2[1-(1-\alpha^2)\gamma_3^2]},$$

$$\lambda^2 F_1 = \frac{-(1-\alpha^2)\gamma_1\gamma_2}{\alpha^2[1-(1-\alpha^2)\gamma_3^2]}.$$

If ω is the velocity of the sun and L and B the coordinates of antapex, we have

$$\begin{aligned} U_g^\circ &= \omega \cos L \cos B, \\ V_g^\circ &= \omega \sin L \cos B, \\ W_g^\circ &= \omega \sin B. \end{aligned}$$

The components $U_0 V_0 W_0$ are then computed by using the direction cosines β_{ij}. Introducing the quantities ν_1 ν_2 and ν_3 by the relations

$$\begin{aligned} \nu_1 &= \beta_{11}\cos L\cos B + \beta_{21}\sin L\cos B + \beta_{31}\sin B, \\ \nu_2 &= \beta_{12}\cos L\cos B + \beta_{22}\sin L\cos B + \beta_{32}\sin B, \\ \nu_3 &= \beta_{13}\cos L\cos B + \beta_{23}\sin L\cos B + \beta_{33}\sin B, \end{aligned} \tag{6}$$

we have

$$U_0 = \omega\nu_1; \quad V_0 = \omega\nu_2; \quad W_0 = \omega\nu_3. \tag{6*}$$

The equations for determining a and b now take the form

$$\left\{ \begin{aligned} \lambda^2 a &= \frac{\alpha^2 + (1-\alpha^2)\gamma_2^2}{\alpha^2[1-(1-\alpha^2)\gamma_3^2]}\cos^2\theta + \frac{\alpha^2 + (1-\alpha^2)\gamma_1^2}{\alpha^2[1-(1-\alpha^2)\gamma_3^2]}\sin^2\theta - \frac{2(1-\alpha^2)\gamma_1\gamma_2}{\alpha^2[1-(1-\alpha^2)\gamma_3^2]}\sin\theta\cos\theta, \\ \frac{\lambda^2 b}{\omega} &= \frac{[\alpha^2+(1-\alpha^2)\gamma_2^2]\nu_1 - (1-\alpha^2)\gamma_1\gamma_2\nu_2}{\alpha^2[1-(1-\alpha^2)\gamma_3^2]}\cos\theta + \frac{[\alpha^2+(1-\alpha^2)\gamma_1^2]\nu_2 - (1-\alpha^2)\gamma_1\gamma_2\nu_1}{\alpha^2[1-(1-\alpha^2)\gamma_3^2]}\sin\theta. \end{aligned} \right. \tag{7}$$

The second members of these equations do not contain the quantities λ and ω and may be tabulated for different squares and different values of θ, if we assume the apex and the vertex direction and the quantity α to be known. Instead of a and b we may use k and l

$$k = \sqrt{a}; \quad l = \frac{b}{\sqrt{a}},$$

so that the function $\Phi(s)$ may be written

$$\Phi(s) = \frac{2 \cdot s}{\sqrt{2\pi}} e^{-\frac{(ks-l)^2}{2}}, \tag{8}$$

if we leave out of consideration a constant factor.

At the end of this paper the quantities

$$K = \frac{1}{\lambda k} \text{ and } L = \frac{\lambda}{\omega} l$$

are found for every square and every tenth degree according to the angle θ. From (7) it is clear that

$$K(\theta + 180°) = K(\theta)$$
$$\text{and } L(\theta + 180°) = -L(\theta).$$

The tables have been made on the following assumptions:—

1. The coordinates of apex are

$$A = 270°, \qquad D = +30°,$$
$$L = 23^{\circ}.50, \qquad B = +21^{\circ}.56;$$

2. The coordinates of vertex are

$$A = 269^{\circ}.70, \qquad D = -17^{\circ}.58,$$
$$L = 340°, \qquad B = 0°;$$

3. As for α I have adopted three different values

$$\alpha = 0.5,\ 0.6,\ 0.7.$$

The coordinates of apex here assumed are most common values. It is more difficult to decide on the values of the coordinates of vertex. As the tables are probably to be used mainly for stars of later spectral type, I have adopted a value which is approximately a mean of the values obtained by CHARLIER and GYLLENBERG[1]) for the stars of spectral type FG and K.

When K and L are found we get $\frac{1}{k}$ and l by multiplying by λ and $\frac{\omega}{\lambda}$ respectively, and then s_0 by the relation (4), which may now be written

$$0 = \frac{1}{s_0} - k^2 s_0 + kl.$$

This equation gives the solution

$$s_0 = \frac{l + \sqrt{l^2 + 4}}{2k}. \tag{9}$$

[1]) L. M. I, 91.

7. *The motions of the stars follow the frequency law of* MAXWELL. If MAXWELL's velocity distribution is fulfilled, we may put

$$A' = B' = C' = \frac{1}{\sigma^2},$$

which gives

$$a = \frac{1}{\sigma^2};\ b = \frac{\omega}{\sigma^2}\cos(\theta - \theta_0),$$

where

$$\theta_0 = \operatorname{arctg}\frac{V_0}{U_0}.$$

Substituting the values of a and b in (4), we get the following equation for determining s_0

$$s_0 = \frac{1}{2}\omega\,\mathrm{Cos}(\theta - \theta_0) + \sqrt{\frac{1}{4}\omega^2\mathrm{Cos}^2(\theta - \theta_0) + \sigma^2}.$$

B. THE CHARACTERISTICS OF THE FUNCTION Φ.

8. The function Φ may be considered a frequency function of the type A, and its moments may be calculated in the following manner.

If $\mu_0, \mu_1, \mu_2 \ldots . \mu_n$ are the moments about the origin, we have

$$\mu_n = \int_0^\infty s^n \Phi\, ds.$$

We take into consideration the following integral

$$A = \frac{2k}{\sqrt{2\pi}} \cdot \int_0^s e^{-\frac{(ks-l)^2}{2}}\, ds. \tag{10}$$

Substituting

$$ks - l = t,$$

we get

$$A = \frac{2}{\sqrt{2\pi}}\int_{-l}^{ks-l} e^{-\frac{t^2}{2}}\, dt = \frac{2}{\sqrt{2\pi}}\int_0^{ks-l} e^{-\frac{t^2}{2}}\, dt - \frac{2}{\sqrt{2\pi}}\int_0^{-l} e^{-\frac{t^2}{2}}\, dt.$$

The function

$$p(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} \tag{11}$$

is the function of probability, and the function

$$P(x) = \int_{-x}^{+x} p(x)\,dx = 2\int_{0}^{x} p(x)\,dx \tag{12}$$

is the integral of probability, which functions are both tabulated.[1]) Now the integral A may be expressed in P. We get

$$A = P(ks - l) - P(-l).$$

As p is an even function, P must be an odd function. Then we have

$$A = P(ks - l) + P(l). \tag{13}$$

When putting $s = +\infty$ we get

$$A(\infty) = 1 + P(l). \tag{14}$$

From (10) we get

$$\frac{\partial A}{\partial l} = \frac{2k}{\sqrt{2\pi}}\int_{0}^{s}(ks - l)\cdot e^{-\frac{(ks-l)^2}{2}}\,ds =$$

$$= \frac{2k^2}{\sqrt{2\pi}}\int_{0}^{s} s\cdot e^{-\frac{(ks-l)^2}{2}}\,ds - \frac{2kl}{\sqrt{2\pi}}\int_{0}^{s} e^{-\frac{(ks-l)^2}{2}}\,ds,$$

thus

$$\frac{2k^2}{\sqrt{2\pi}}\int_{0}^{s} s\cdot e^{-\frac{(ks-l)^2}{2}}\,ds = \frac{\partial A}{\partial l} + l\cdot A.$$

From (13) we get

$$\frac{\partial A}{\partial l} = -P'(ks - l) + P'(l).$$

From the definition of $P(x)$ and $p(x)$ we have

$$P'(x) = 2\,p(x),\quad p'(x) = -x\cdot p(x), \tag{15}$$

in general

$$P^{(r)}(x) = 2\,p^{(r-1)}\,x.$$

Then we get the formula

$$\frac{2k^2}{\sqrt{2\pi}}\int_{0}^{s} s\cdot e^{-\frac{(ks-l)^2}{2}}\,ds = -2\,p(ks - l) + 2\,p(l) + l\,P(ks - l) + l\cdot P(l). \tag{16}$$

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[1]) CHARLIER: "Die Grundzüge der Mathematischen Statistik". CZUBER: "Wahrscheinlichkeitsrechnung", where $P(x\sqrt{2})$ is tabulated with 7 decimals.

We put $A_0 = A(\infty)$. Then we have

$$A_0 = \frac{2k}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{(ks-l)^2}{2}} ds. \tag{17}$$

By repeated derivations we get the following equations

$$\frac{\partial A_0}{\partial l} = \frac{2k}{\sqrt{2\pi}} \int_0^\infty (ks-l) \cdot e^{-\frac{(ks-l)^2}{2}} ds,$$

$$\frac{\partial^2 A_0}{\partial l^2} + A_0 = \frac{2k}{\sqrt{2\pi}} \int_0^\infty (ks-l)^2 \cdot e^{-\frac{(ks-l)^2}{2}} ds,$$

$$\frac{\partial^3 A_0}{\partial l^3} + 3\frac{\partial A_0}{\partial l} = \frac{2k}{\sqrt{2\pi}} \int_0^\infty (ks-l)^3 \cdot e^{-\frac{(ks-l)^2}{2}} ds.$$

The moments may now be expressed in A_0 and the derived functions of A_0.

$$\left\{\begin{aligned}
\mu_0 &= \int_0^\infty \Phi \, ds = \frac{1}{k^2}\left[\frac{\partial A_0}{\partial l} + l A_0\right], \\
\mu_1 &= \int_0^\infty s \cdot \Phi \, ds = \frac{1}{k^3}\left[\frac{\partial^2 A_0}{\partial l^2} + 2l\frac{\partial A_0}{\partial l} + (1+l^2) A_0\right], \\
\mu_2 &= \int_0^\infty s^2 \Phi \, ds = \frac{1}{k^4}\left[\frac{\partial^3 A_0}{\partial l^3} + 3l\frac{\partial^2 A_0}{\partial l^2} + 3(1+l^2)\frac{\partial A_0}{\partial l} + l(3+l^2) A_0\right].
\end{aligned}\right. \tag{18}$$

Using the equations (14) and (15), we get A_0 and the derived functions of A_0 expressed in P and p

$$\begin{aligned}
A_0 &= 1 + P(l), \\
\frac{\partial A_0}{\partial l} &= 2p(l), \\
\frac{\partial^2 A_0}{\partial l^2} &= -2lp(l), \\
\frac{\partial^3 A_0}{\partial l^3} &= 2l^2 p(l) - 2p(l).
\end{aligned}$$

The equations (18) now take the form

$$\left\{\begin{aligned}
\mu_0 &= \frac{1}{k^2}[l + l P(l) + 2p(l)], \\
\mu_1 &= \frac{1}{k^3}[1 + l^2)\ (1 + P(l)) + 2lp(l)], \\
\mu_2 &= \frac{1}{k^4}[l(3 + l^2)\ (1 + P(l)) + 4 \cdot p(l) + 2l^2 p(l)].
\end{aligned}\right. \tag{19}$$

9. Let us now consider how the moments of higher order may be derived from those of lower order.

We have

$$\mu_n = \int_0^\infty s^n \, \Phi \, ds.$$

From this equation we get by derivation

$$\frac{\partial \mu_n}{\partial l} = \int_0^\infty s^n \, (ks - l) \, \Phi \, ds = k \int_0^\infty s^{n+1} \, \Phi \, ds - l \int_0^\infty s^n \, \Phi \, ds = k \, \mu_{n+1} - l \, \mu_n,$$

and consequently

$$k \, \mu_{n+1} = \frac{\partial \mu_n}{\partial l} + l \, \mu_n. \tag{20}$$

We put

$$u_0 = 1 + P(l),$$
$$u_1 = k^2 \, \mu_0,$$
$$u_2 = k^3 \, \mu_1 \text{ and so on.}$$

From (19) we thus get

$$u_1 = l \, u_0 + 2 \, p \, (l),$$
$$u_2 = l \, u_1 + u_0,$$
$$u_3 = l \, u_2 + 2 \, u_1.$$

Let us suppose that the law

$$u_{n+1} = l \, u_n + n \, u_{n-1} \tag{21}$$

is true for all values of n up to a certain limit.

From (20) we have

$$u_{n+1} = \frac{\partial \, u_n}{\partial \, l} + l \, u_n.$$

The combination of this equation with (21) gives

$$\frac{\partial \, u_n}{\partial \, l} = n \, u_{n-1}. \tag{22}$$

Further we get from (21)

$$\frac{\partial \, u_{n+1}}{\partial \, l} = l \, \frac{\partial \, u_n}{\partial \, l} + u_n + n \, \frac{\partial \, u_{n-1}}{\partial \, l}$$

and according to (22) and (21)

$$\frac{\partial \, u_{n+1}}{\partial \, l} = (n + 1) \, u_n. \tag{23}$$

The formula (20) is true for every value of n; thus we have

$$u_{n+2} = \frac{\partial \, u_{n+1}}{\partial \, l} + l \, u_{n+1}$$

and according to (23) we get

$$u_{n+2} = l\,u_{n+1} + (n+1)\,u_n.$$

Hence the law

$$u_{n+1} = l\,u_n + n\,u_{n-1}$$

must be universally true.

It is easy to show that

$$\begin{aligned} u_0(l) + u_0(-l) &= 2, \\ u_1(l) - u_1(-l) &= 2\,l, \\ u_2(l) + u_2(-l) &= 2\,l^2 + 2, \\ u_3(l) - u_3(-l) &= 2\,l^3 + 6\,l, \end{aligned}$$

and if we put

$$q_n(l) = u_n(l) + (-1)^n\,u_n(-l),$$

we have in general

$$q_{n+1} = l\,q_n + n\,q_{n-1}.$$

The last mentioned formulae may conveniently be used as formulae for checking the numerical calculation of the functions u_n.

10. *The mean value* of s is defined by the equation

$$\bar{s} = \int_0^\infty s\,\Phi\,ds : \int_0^\infty \Phi\,ds,$$

thus

$$\bar{s} = \mu_1 : \mu_0,$$

and from the definition of u_i

$$k\bar{s} = \frac{u_2}{u_1}.$$

The mode is the value of s which corresponds to the maximum of Φ. By taking the logarithm of Φ and then deriving, we get the equation for determining the mode. If the mode is denoted s_0, we get

$$s_0 = \frac{l + \sqrt{l^2+4}}{2k}.$$

The median is defined by the equation

$$\int_0^s \Phi\,ds = \int_s^\infty \Phi\,ds = \frac{1}{2}\int_0^\infty \Phi\,ds.$$

For the calculation of this quantity see (24) page 23.

The harmonical mean is computed from the mean value of the inverse value of s. If the harmonical mean is denoted H, we have

$$\frac{1}{H} = \int_0^\infty \frac{1}{s}\,\Phi\,ds : \int_0^\infty \Phi\,ds.$$

Using (14) and (19), we get

$$H = \frac{u_1}{k \cdot u_0}.$$

The dispersion about a value s_0 is computed from the equation

$$\sigma^2 = \int_0^\infty (s - s_0)^2 \Phi \, ds : \int_0^\infty \Phi \, ds = \left[\int_0^\infty s^2 \Phi \, ds - 2 s_0 \int_0^\infty s \Phi \, ds + s_0^2 \int_0^\infty \Phi \, ds \right] : \int_0^\infty \Phi \, ds,$$

thus

$$\sigma^2 = \frac{\mu_2 - 2 s_0 \mu_1 + s_0^2 \mu_0}{\mu_0}.$$

If we put $s_0 = \bar{s}$, we get

$$k^2 \sigma^2 = \frac{u_3}{u_1} - \left(\frac{u_2}{u_1} \right)^2.$$

If the frequency-function of s is very asymmetrical, it seems to be more advantageous to use an upper and a lower limit instead of the dispersion.

We compute two quantities s_1 and s_0 so that the equations

$$\int_0^{s_1} \Phi \, ds = \frac{1}{6} \int_0^\infty \Phi \, ds \quad \text{and} \quad \int_{s_2}^\infty \Phi \, ds = \frac{5}{6} \int_0^\infty \Phi \, ds$$

are fulfilled. Thus if we use s_1 and s_2 as the upper and the lower limit,[1]) $\frac{2}{3}$ of the material lies between these limits. Using the formula (16) these equations may be written

$$-2p(ks - l) + 2p(l) + l P(ks - l) + l P(l) = n[l + l P(l) + 2p(l)],$$

$$n = \frac{1}{6}, \frac{5}{6}$$

or

$$2p(ks - l) - l P(ks - l) = (1 - n)[2p(l) + l P(l)] - n l.$$

Putting

$$2p(x) - l P(x) = Q_l(x)$$

we get

$$Q_l(ks - l) = (1 - n) Q_{-l}(l) - n l,$$

and if Q^{-1} is the inverse function of Q, we get

$$ks = Q_l^{-1}[(1 - n) Q_{-l}(l) - n l] + l, \tag{24}$$

where the second member is a function of l and n only. When putting $n = \frac{1}{6}, \frac{1}{5}, \frac{5}{6}$

[1]) The terms upper and lower limit are here taken in a relative sense.

respectively, the lower limit, the median and the upper limit can be calculated from this equation.

Skewness and excess. From the absolute moments μ_s about the origin we get the relative moments ν_s' by the relations

$$\nu_s' = \mu_s : \mu_0 = \frac{1}{k^s} \cdot \frac{u_{s+1}}{u_1},$$

then the relative moments about the mean[1])

$$\begin{aligned} \sigma^2 = \nu_2 &= \nu_2' - b^2, \\ \nu_3 &= \nu_3' - 3\,b\,\nu_2' + 2\,b^3, \\ \nu_4 &= \nu_4' - 4\,b\,\nu_3' + 6\,b^2\,\nu_2' - 3\,b^4, \end{aligned}$$

where $b = \nu_1'$.

From these equations we compute the characteristics β_3 and β_4:—

$$\begin{aligned} \beta_3 &= -\nu_3 : 6\,\sigma^3, \\ \beta_4 &= \frac{1}{24}\,(\nu_4 : \sigma^4 - 3), \end{aligned}$$

and finally we get the skewness S and the excess E by the relations

$$\begin{aligned} S &= 3\,\beta_3, \\ E &= 3\,\beta_4. \end{aligned}$$

11. In figs. 1, 2 and 3 I give a graphical representation of the function Φ according to different values given to k and l.[2])

If MAXWELL's velocity distribution is fulfilled, we may take into consideration the following cases according to the dispersion

$$\begin{aligned} &\text{I} \quad \sigma = \omega, \\ &\text{II} \quad \sigma = 0.5\,\omega, \\ &\text{III} \quad \sigma = 2\,\omega. \end{aligned}$$

With reference to the direction $\theta - \theta_0$ we make the following assumptions

$$\begin{aligned} &\text{a)} \quad \theta - \theta_0 = \quad 0^\circ, \\ &\text{b)} \quad \theta - \theta_0 = \quad 90^\circ, \\ &\text{c)} \quad \theta - \theta_0 = 180^\circ. \end{aligned}$$

The values of k and l corresponding to these nine cases are computed from

$$l = \frac{b}{\sqrt{a}} \text{ and } k = \sqrt{a},$$

where

$$a = \frac{1}{\sigma^2}; \quad b = \frac{\omega}{\sigma^2} \cos(\theta - \theta_0).$$

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[1]) CHARLIER: "Researches into the theory of probability", L. M. II, 4.

[2]) To get the curves more comparable, the function Φ has been divided by the whole area $\int_0^\infty \Phi\, ds$.

TABLE III.

	Ia	Ib	Ic	IIa	IIb	IIc	IIIa	IIIb	IIIc
k	1	1	1	0.5	0.5	0.5	2	2	2
l	+1	0	−1	+0.5	0	−0.5	+2	0	−2
$\overline{s}$	1.78	1.25	0.90	2.98	2.51	2.12	1.24	0.63	0.34
m	1.72	1.18	0.82	2.68	2.26	1.86	1.13	0.50	0.25
s_0	1.62	1.00	0.62	2.56	2.00	1.56	1.21	0.50	0.21
H	1.29	0.80	0.53	2.02	1.60	1.28	1.03	0.40	0.02
σ	0.79	0.66	0.53	1.31	1.33	1.18	0.45	0.33	0.22
S	−0.20	−0.31	−0.38	−0.26	−0.31	−0.36	−0.11	−0.31	−0.25
E	−0.005	+0.031	+0.120	+0.006	+0.031	+0.120	−0.021	+0.031	+0.584
s_1	0.99	0.60	0.40	1.54	1.21	0.96	0.80	0.30	0.13
s_2	2.55	1.89	1.41	4.40	3.79	3.26	1.68	0.95	0.54
$\frac{\sigma}{\overline{s}}$	0.44	0.58	0.59	0.44	0.58	0.56	0.36	0.53	0.65

The table III contains the characteristics of the function Φ. The most easily obtained quantity is the mode. It corresponds to the maximum frequency and is to be considered as the most probable value of s. As is to be seen from the table, its deviation from the mean value may in certain cases be very great, but in the cases treated here it does not exceed 30 per cent of the mean value. The dispersion varies considerably in size, but does not vary so much with the direction. The skewness is always negative, which is also seen from the figures. The excess, on the other hand, may be both positive and negative and has in certain cases a very considerable positive value.

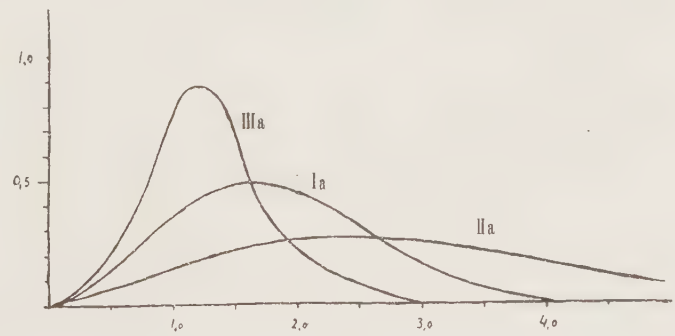

Fig. 1.

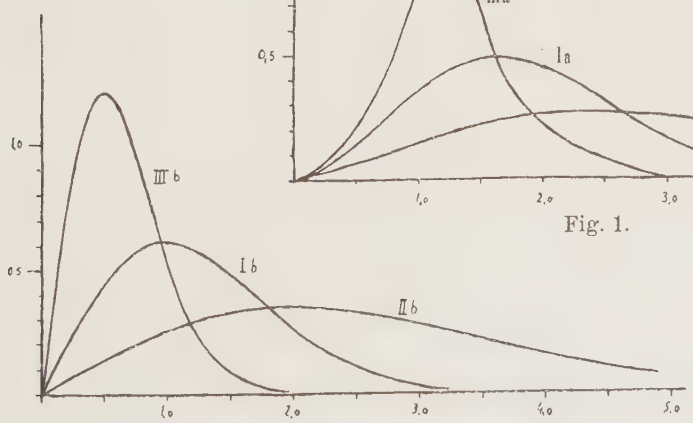

Fig. 2.

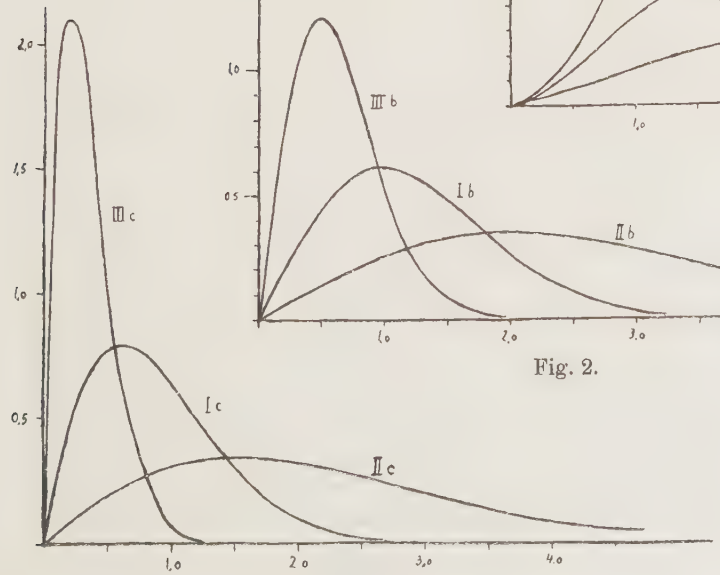

Fig. 3.

In computing the parallax of a star we may use the formula

$$\pi = \frac{\mu}{s}.$$

s denotes the linear proper motion and μ the total amount of the proper motion. The parallax is then computed by substituting any one of the quantities s_0, $\bar{s}$, H and m for s.

How great will the mean error of π now be, if π is computed in this way?

We have

$$\varepsilon^2(\pi) = \left(\frac{\partial \pi}{\partial s}\right)^2 \varepsilon^2(s),$$

thus

$$\varepsilon(\pi) = \frac{\mu}{s^2}\varepsilon(s)$$

on the supposition that the mean error of μ may be neglected.

The formula may be written

$$\frac{\varepsilon(\pi)}{\pi} = \frac{\varepsilon(s)}{s}.$$

Substituting σ for $\varepsilon(s)$ and $\bar{s}$ for s we find from the table that $\frac{\varepsilon(\pi)}{\pi}$ has a value about 0.5.

The absolute magnitude M of a star may be computed from the formula

$$M = m - \frac{1}{b}{}^e\log s + \frac{1}{b}{}^e\log \mu \quad (b = 0.2 \times \text{Mod.} = 0.46052), \tag{25}$$

where for instance $\bar{s}$ may be substituted for s. The mean error of M we get from

$$\varepsilon(M) = \frac{1}{bs}\varepsilon(s).$$

Using the same value of $\frac{\varepsilon(s)}{s}$ as before, we find that $\varepsilon(M)$ is about 1^m.

The mean error of M may also be computed from the upper and the lower limit, if we put

$$2 \cdot \varepsilon(M) = \frac{1}{b}{}^e\log \frac{s_2}{s_1}.$$

The table below contains $\varepsilon(M)$ computed from this formula

THE MEAN ERROR OF THE COMPUTED VALUE OF M.

l	$\varepsilon(M)$	l	$\varepsilon(M)$	l	$\varepsilon(M)$	l	$\varepsilon(M)$	l	$\varepsilon(M)$	l	$\varepsilon(M)$
−3.0	1.52	−2.0	1.47	−1.0	1.38	+0.0	1.24	+1.0	1.02	+2.0	0.80
−2.9	1.52	−1.9	1.47	−0.9	1.37	+0.1	1.22	+1.1	1.00	+2.1	0.78
−2.8	1.52	−1.8	1.46	−0.8	1.36	+0.2	1.20	+1.2	0.98	+2.2	0.76
−2.7	1.51	−1.7	1.45	−0.7	1.35	+0.3	1.18	+1.3	0.95	+2.3	0.74
−2.6	1.51	−1.6	1.44	−0.6	1.34	+0.4	1.16	+1.4	0.93	+2.4	0.72
−2.5	1.50	−1.5	1.43	−0.5	1.32	+0.5	1.14	+1.5	0.91	+2.5	0.70
−2.4	1.50	−1.4	1.42	−0.4	1.31	+0.6	1.12	+1.6	0.89	+2.6	0.69
−2.3	1.49	−1.3	1.42	−0.3	1.29	+0.7	1.09	+1.7	0.87	+2.7	0.67
−2.2	1.49	−1.2	1.41	−0.2	1.27	+0.8	1.07	+1.8	0.85	+2.8	0.65
−2.1	1.48	−1.1	1.40	−0.1	1.26	+0.9	1.04	+1.9	0.82	+2.9	0.63

It is, however, not strictly correct to use the mean value of s in the formula (25), as in this formula we have log s.

It would be better to use the mean value of $\log s$, but as this is difficult to find, we may use the mode of $\log s$.

To obtain this we substitute

$$x = \frac{1}{b} \log s,$$

$$dx = \frac{ds}{bs}$$

in the frequency function of s (8). We thus get the frequency function of x

$$\Phi_1(x)\, dx = \frac{2b\, dx}{\sqrt{2\pi}} e^{2bx} \cdot e^{-\frac{(ke^{bx} - l)^2}{2}}.$$

By taking the logarithm of Φ_1 and deriving, we get the equation for computing the mode

$$2 = (k \cdot e^{bx} - l)\, k \cdot e^{bx},$$

which gives the solution

$$x = 5 \cdot {}^{10}\log \frac{l + \sqrt{l^2 + 8}}{2k}.$$

Thus the formula for computing the absolute magnitude is

$$M = m - 5\, {}^{10}\log \frac{l + \sqrt{l^2 + 8}}{2k} + 5 \cdot {}^{10}\log \mu. \tag{26}$$

We find that, if we use the mode of $\log s$ instead of the mode of s, we always get a lower value of M, and the difference is

$$\Delta M = 5\, {}^{10}\log \frac{l + \sqrt{l^2 + 8}}{l + \sqrt{l^2 + 4}}. \tag{26*}$$

In fig. 4 I give a graphical representation of the function $\Phi_1(x)$. The dotted lines correspond to the maximum of $\Phi(s)$.

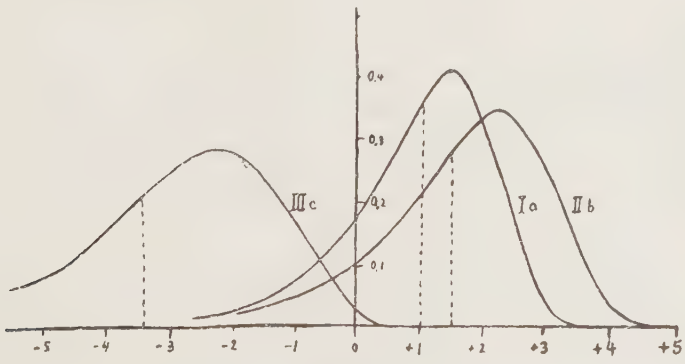

Fig. 4.

C. THE DETERMINATION OF THE VELOCITY ELLIPSOID FROM THE RADIAL VELOCITIES UNDER SPECIAL PRESUMPTIONS.

12. To be able to make use of the tables at the end of this study it is necessary to know the dimensions of the velocity ellipsoid λ and $\alpha\lambda$ on the supposition that the distribution is an ellipsoid of rotation. As a rule, no ellipsoid of rotation is obtained, if the velocity distribution is derived in a general way, but as different results are ordinarily obtained for the two shorter axes according as we proceed from the proper motions or the radial velocities (see chapter I), we may be entitled, in a case like this, to presume that the distribution is an ellipsoid of rotation.

If the axes of the ellipsoid $\alpha_1\ \alpha_2\ \alpha_3$ have been derived in the ordinary way and we assume α_1 to be the longest of the axes, we may put

$$\lambda = \alpha_1 \text{ and } \alpha\lambda = \frac{\alpha_2 + \alpha_3}{2}.$$

However, in drawing up the tables, I have had to fix a certain vertex direction. The dimensions of the ellipsoid (α and λ) may now be derived from the radial velocities, provided that the apex vertex and the velocity of the sun are known.

13. In the system K_0 the velocity ellipsoid has the form (2)

$$f = A(U-U_0)^2 + B(V-V_0)^2 + C(W-W_0)^2 + \\ + 2D(V-V_0)(W-W_0) + 2E(U-U_0)(W-W_0) + 2F(U-U_0)(V-V_0).$$

If we integrate over U between $-\infty$ and $+\infty$, we get

$$\int_{-\infty}^{+\infty} F\,dU\,H_1\,e^{-\frac{1}{2}f_1},$$

where

$$f_1 = B_1(V-V_0)^2 + C_1(W-W_0)^2 + 2D_1(V-V_0)(U-U_0)$$

and

$$AB_1 = BA - F^2,$$
$$AC_1 = CA - E^2,$$
$$AD_1 = DA - EF.$$

The quantities H_1, f_1 and B_1 have, of course, not the same values as before (page 14), as we have here integrated over U instead of W.

Then we integrate over V between $+\infty$ and $-\infty$. We thus get a frequency function of W only

$$F(W) = H_2\,e^{-\frac{1}{2}f_2},$$

where

$$f_2 = C_2(W-W_0)^2$$

and

$$C_2 B_1 = C_1 B_1 - D_1^2.$$

The relations connecting $B_1\, C_1\, D_1$ with $A'B'C'$ are, if we put $A' = B'$ and make use of (5),

$$(27)\quad \begin{cases} B_1 A = B'C'(1-\gamma_3^2) + B'^2\gamma_3^2, \\ C_1 A = B'C'(1-\gamma_2^2) + B'^2\gamma_2^2, \\ D_1 A = -(B'^2 - B'C')\gamma_2\gamma_3, \\ A = B'(1-\gamma_1^2) + C'\gamma_1^2. \end{cases}$$

Supposing that the axes of the ellipsoid are λ and $\alpha\lambda$, we have

$$C' = \frac{1}{\lambda^2};\quad B' = \frac{1}{\alpha^2\lambda^2}.$$

The equations (27) now take the form

$$\lambda^2 B_1 = \frac{\alpha^2 + (1-\alpha^2)\gamma_3^2}{\alpha^2[1-(1-\alpha^2)\gamma_1^2]},$$

$$\lambda^2 C_1 = \frac{\alpha^2 + (1-\alpha^2)\gamma_2^2}{\alpha^2[1-(1-\alpha^2)\gamma_1^2]},$$

$$\lambda^2 D_1 = \frac{-(1-\alpha^2)\gamma_2\gamma_3}{\alpha^2[1-(1-\alpha^2)\gamma_1^2]}.$$

Using these equations, we get

$$\lambda^2 C_2 = \frac{1}{\alpha^2 + (1-\alpha^2)\gamma_3^2}.$$

If we put $C_2 = \frac{1}{\varrho^2}$, we get

$$(28)\qquad \varrho^2 = \lambda^2\alpha^2 + \lambda^2(1-\alpha^2)\gamma_3^2.$$

Here ϱ denotes the dispersion in the line of sight, and this quantity is to be computed from the radial velocities for each square. If n is the number of stars in a square, we have

$$\varrho^2 = \frac{1}{n}\Sigma(W-W_0)^2.$$

W_0 is the component of stream velocity in the line of sight and can be computed when the apex and the velocity of the sun are known. γ_3 is the angle which the radius vector to the centre of the square makes with the vertex direction. When ϱ is computed we get 48 equations (28), and from these the unknown quantities $\alpha^2\lambda^2$ and $(1-\alpha^2)\lambda^2$ may be computed by means of least squares.

If we assume α to be known, the quantity λ may be determined in the following manner. We have

$$\lambda^2\alpha^2 + \lambda^2(1-\alpha^2)\gamma_3^2 = \frac{1}{n}\Sigma^1(W-W_0)^2,$$

or

$$n\lambda^2\alpha^2 + n\lambda^2(1-\alpha^2)\gamma_3^2 = \Sigma^1(W-W_0)^2,$$

where n is the number of stars with known radial velocities in a square and Σ^1 is to be taken over these n stars. By adding the 48 equations we get

$$N\lambda^2\alpha^2 + \lambda^2(1-\alpha^2)\Sigma_1 n\gamma_s^2 = \Sigma(W-W_0)^2,$$

or

$$\lambda^2 = \frac{\Sigma(W-W_0)^2}{N\alpha^2 + (1-\alpha^2)\Sigma_1 n\gamma_s^2}, \tag{29}$$

where N is the total number of stars with known radial velocities and Σ is to be taken over the whole heavens, Σ_1 over the 48 squares. The formula (29) has the advantage of being available when the material is small.

TABLE IV.

Square	γ_s	ν_s	Square	γ_s	ν_s
GA_1	− 0.0594	− 0.4263	GF_1	− 0.0594	+ 0.2975
GA_2	+ 0.0594	− 0.2975	GF_2	+ 0.0594	+ 0.4263
GB_1	+ 0.5563	− 0.9138	GE_1	+ 0.5563	− 0.3932
GB_2	+ 0.1946	− 0.8259	GE_2	+ 0.1946	− 0.3053
GB_3	− 0.2414	− 0.5221	GE_3	− 0.2414	− 0.0014
GB_4	− 0.5852	− 0.1183	GE_4	− 0.5852	+ 0.4023
GB_5	− 0.7055	+ 0.2314	GE_5	− 0.7055	+ 0.7520
GB_6	− 0.5563	+ 0.3932	GE_6	− 0.5563	+ 0.9138
GB_7	− 0.1946	+ 0.3053	GE_7	− 0.1946	+ 0.8259
GB_8	+ 0.2414	+ 0.0014	GE_8	+ 0.2414	+ 0.5221
GB_9	+ 0.5852	− 0.4023	GE_9	+ 0.5852	+ 0.1183
GB_{10}	+ 0.7055	− 0.7520	GE_{10}	+ 0.7055	− 0.2314
GC_1	+ 0.7932	− 0.9825	GD_1	+ 0.7932	− 0.7987
GC_2	+ 0.4092	− 0.9298	GD_2	+ 0.4092	− 0.7460
GC_3	− 0.0844	− 0.6525	GD_3	− 0.0844	− 0.4687
GC_4	− 0.5554	− 0.2250	GD_4	− 0.5554	− 0.0413
GC_5	− 0.8776	+ 0.2381	GD_5	− 0.8776	+ 0.4219
GC_6	− 0.9646	+ 0.6128	GD_6	− 0.9646	+ 0.7966
GC_7	− 0.7932	+ 0.7987	GD_7	− 0.7932	+ 0.9825
GC_8	− 0.4092	+ 0.7460	GD_8	− 0.4092	+ 0.9298
GC_9	+ 0.0844	+ 0.4687	GD_9	+ 0.0844	+ 0.6525
GC_{10}	+ 0.5554	+ 0.0413	GD_{10}	+ 0.5554	+ 0.2250
GC_{11}	+ 0.8776	− 0.4219	GD_{11}	+ 0.8776	− 0.2381
GC_{12}	+ 0.9646	− 0.7966	GD_{12}	+ 0.9646	− 0.6128

Table IV contains the quantities ν_s and γ_s for each square. The table was made on the same assumption as before concerning the apex and vertex (page 17). W_0 is obtained from

$$W_0 = \nu_s\omega,$$

where ω is the velocity of the sun.

14. CHARLIER and GYLLENBERG[1]) have computed the velocity ellipsoid for stars of different spectral type. From their investigation we get the values of the axes $\alpha_1\ \alpha_2\ \alpha_3$ ($q' = 0.75$):—

[1]) L. M. I, 91.

Spectral-type	a_1	a_2	a_3	a
B	3.10	1.86	0.50	0.38
A	5.04	2.73	1.99	0.47
F	5.20	3.71	2.87	0.63
G	7.11	5.10	3.04	0.57
K	5.86	3.80	3.16	0.59
M	4.99	2.37	0.97	0.33

The quantity α in the last column is computed from

$$\alpha = \frac{\alpha_2 + \alpha_3}{2\,\alpha_1}.$$

We find that α has much the same value for stars of the spectral type *F*, *G* and *K*. The mean value of α for these stars is 0.60. Using 0.60 as the value of α, I have computed the quantity λ from 197 stars of spectral type G, brighter than $6^{m}_{.}0$, with known radial velocities. As regard the velocity of the sun I have adopted the value 3.960 sir./st. obtained by GYLLENBERG. We find from table IV that γ_3^2 has the same value for groups of four squares each. I have therefore put together four and four squares and computed the value of ϱ for such a group. I have rejected those stars whose value of $(W - W_0)$ exceeded three times the dispersion ϱ. The result is

(30) $$\lambda = 5.654 \pm 0.408.$$

CHAPTER III.

A. STARS BRIGHTER THAN $6^{m}.0$.

15. The method for determining the absolute magnitude I have applied to the stars of spectral type G brighter than 6^{m}. The proper motions I have taken from BOSS' catalogue and the spectrum and visual magnitude from Harvard 50. As for the Apex, Vertex and the solar motion I have made the same assumption as before (page 17). From the proper motions $\Delta l \cos b$ and Δb I have computed the quantities θ and μ by the relations

$$\Delta l \cos b = \mu \cos \theta,$$
$$\Delta b = \mu \sin \theta,$$

and then with the help of the tables at the end of this investigation I have by means of interpolation determined the quantities l and $\frac{1}{k}$, assuming $\alpha = 0.60$ and $\lambda = 5.654$. The absolute magnitude M is then obtained from the formula

$$M = m - 5 \log \frac{l + \sqrt{l^2 + 8}}{2k} + 5 \log \mu.$$

The material consists of 415 stars, and this material I have divided into galactic stars (b within $\pm 30°$) and non-galactic stars. Further I have divided the material into G_0, G_2 and G_5 stars (a few stars of the spectral type G_8 being taken together with the G_5 stars), and finally a division has been made with regard to the apparent magnitude. The result is shown in the table V.

TABLE V.

FREQUENCY DISTRIBUTION $F(M)$ OF THE ABSOLUTE MAGNITUDE M.

M	All stars	G_0	G_2	G_5	$\leq 4^{m}.99$	$5^{m}.00$–$5^{m}.64$	$5^{m}.65$–$5^{m}.99$	Gal. stars	Non-gal. stars
−9 −8	2	1		1			2	1	1
−8 −7	9	4		5	6	2	1	6	3
−7 −6	10	8		2	7	3		7	3
−6 −5	20	10		10	7	7	6	13	7
−5 −4	33	18		15	23	4	6	22	11
−4 −3	45	15	6	24	19	12	14	25	20
−3 −2	64	21	8	35	17	30	17	43	21
−2 −1	62	28	6	28	11	23	28	28	34
−1 0	47	12	7	28	12	14	21	21	26
0 +1	46	21	2	23	8	18	20	22	24
+1 +2	24	13	3	8	5	13	6	16	8
+2 +3	27	17	2	8	10	8	9	10	17
+3 +4	8	7		1	3	4	1	4	4
+4 +5	9	7		2	1	5	3	4	5
+5 +6	6	3	1	2	2	1	3	3	3
+6 +7	2	1		1	1	1		1	1
+7 +8	1			1			1		1
N	415	186	35	194	132	145	138	226	189

FREQUENCY DISTRIBUTION OF THE ABSOLUTE MAGNITUDE.
ALL STARS BRIGHTER THAN $6^{m}.0$.

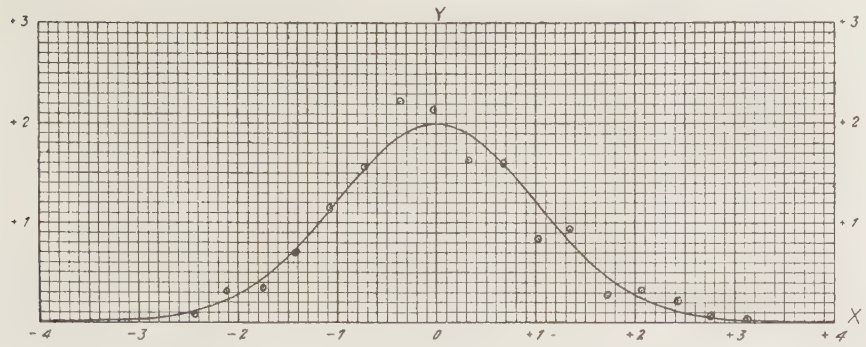

Fig. 5.

$$X = \frac{M - \overline{M}}{\sigma} \qquad Y = \frac{5\sigma}{N} F(M)$$

16. In examining the frequency distribution we find no division into two separate classes. One irregularity or other may just as well have been caused by a mere chance. However, this is no proof whatever that there may not be in reality giants and dwarfs. The uncertainty of the determination of the absolute magnitude by means of this method is so great that it might very well reduce the frequency curve into a fairly normal distribution if the means are not too far apart. The material is too scanty to allow of an examination whether there is any negative excess. A large negative excess would be indicative of the existence of two classes of magnitude.

The means and the dispersions have been put together in table VI.

TABLE VI.

THE MEANS AND DISPERSIONS OF THE ABSOLUTE MAGNITUDE.

	Mean	Dispersion
All stars	-1.43 ± 0.141	2.88 ± 0.100
G_0	-1.23 ± 0.246	3.85 ± 0.174
G_2	-1.07 ± 0.348	2.06 ± 0.246
G_5	-1.69 ± 0.190	2.65 ± 0.134
≤ 4.99	-2.24 ± 0.268	3.08 ± 0.189
$5^{m}.00$–$5^{m}.64$	-1.05 ± 0.225	2.71 ± 0.159
$5^{m}.65$–$5^{m}.99$	-1.06 ± 0.228	2.68 ± 0.161
Gal. stars	-1.78 ± 0.189	2.85 ± 0.134
Non-gal. stars	-1.01 ± 0.207	2.85 ± 0.146

It is to be seen that the G_5 stars are a little brighter than G_0 and G_2. That the mean and the dispersion for G_0 occupy an intermediate position between G_2 and G_5 may probably be due to the fact that the stars for which the subclass was difficult to determine have been referred to G. The difference in absolute magnitude is greater for

stars of different apparent magnitude than for stars of different subclass. The apparently brighter stars are also, as a rule, of greater absolute luminosity. Further the stars that lie in the Milky Way are brighter than those surrounding its poles. There is reason to expect that the stars with very great absolute luminosity which are visible at a distance exceeding the extension of the Milky Way in the direction of its poles, will appear more numerously about the galactic plane than about the poles. To this fact may probably be due the result here obtained. The dispersion seems to be greater for the apparently brighter stars.

17. The dispersion here computed is of course not the true dispersion, as the uncertainty of the determination of M also affects the result. If the mean error ε in the determination of M is known, we can compute the true dispersion σ by means of the formula

$$\sigma = \sqrt{\sigma_1^2 - \varepsilon^2},$$

where σ_1 is the dispersion obtained from the computed values of M. The mean error $\varepsilon(M)$ for a single star is obtained from the formula (see page 26)

$$2 \cdot \varepsilon(M) = 5 \log \frac{s_2}{s_1}.$$

I have determined $\varepsilon(M)$ for each star with the help of this formula and then computed the mean value of $\varepsilon^2(M)$. I got the following result

$$\frac{\Sigma\, \varepsilon^2(M)}{N} = 1.^{m}37.$$

Putting this value equal to ε^2 in the above formula and using the value of σ_1 for all stars we get

$$\sigma = \sqrt{(2.^{m}88)^2 - 1.^{m}37} = 2.^{m}63.$$

We have not yet made allowance for the possibility of different velocities for giant and dwarf stars. If the velocities of dwarfs are greater than the velocities of giants, this circumstance would affect the result in such a manner that the brighter stars would appear more bright and the fainter stars more faint. This question is treated in the following part of this chapter.

18. The computed mean value of the absolute magnitude is not the mean value for all stars of spectral type G in our universe. It is a mean value for stars brighter than 6^{m}. It is the same thing with the dispersion. We may, according to MALMQUIST[1]), denote the mean value of the absolute magnitude for stars brighter than the apparent magnitude m by $\overline{M}_{-\infty, m}$.

If we desire to know the mean value M_0 and the dispersion σ for all stars, it is necessary to know the number of stars $A(m)$ brighter than a given apparent magnitude. From Harvard 50 we get

[1]) L. M. II, 22.

TABLE VII.

NUMBER OF STARS BRIGHTER THAN m.

m	$A(m)$ observed	$A(m)$ computed	Δ
3.0	9	10	— 1
3.5	22	19	+ 3
4.0	33	36	— 3
4.5	71	67	+ 4
5.0	134	127	+ 7
5.5	236	240	— 4
6.0	457	454	+ 3

The second column contains the observed number of $A(m)$ for different m. The numbers in the third column are computed from the formula

$$^{10}\log A(m) = 0.552\, m - 0.655. \tag{31}$$

The fourth column contains the difference between the observed and the computed values of $A(m)$. We see that the BOSS catalogue is not complete down to the magnitude $6^m.0$. As limiting magnitude I have therefore adopted the value $5^m.98$. This value corresponds to the number 415 in the equation (31).

Supposing that the frequency distribution $\varphi(M)$ of the absolute magnitude M is of the form

$$\varphi(M) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(M-M_0)^2}{2\sigma^2}},$$

MALMQUIST has given a relation between $\overline{M}_{-\infty, m_1}$, M_0 and σ, and between $\sigma_{-\infty, m_1}$ and σ

$$\overline{M}_{-\infty, m_1} = M_0 - \sigma^2 \left[\frac{d \log A(m)}{d m}\right]_{m = m_1}, \tag{32}$$

$$\sigma^2_{-\infty, m_1} = \sigma^2 + \sigma^4 \frac{d^2 \log A(m)}{d m^2}. \tag{33}$$

Using the formula (32) we get for all G-stars brighter than the magnitude $5^m.98$

$$-1.43 = M_0 - \sigma^2 \cdot 1.271. \tag{34}$$

According to (31) and (33) we may put $\sigma_{-\infty, m} = \sigma$. From the value of σ found above we thus get

$$M_0 = +7.89.$$

We will now try to find a new relation between M_0 and σ. This may be done by using the mean value of $\frac{1}{R}$ obtained in the first chapter and then applying MALMQUIST's formula

$$\left(\frac{\overline{1}}{R}\right)_{-\infty, m} = e^{b M_0 + \frac{1}{2} b^2 \sigma^2} \frac{A(m - b\sigma^2)}{A(m)}. \tag{35}$$

But as this mean value is, according to what we have found, very sensible of an exclusion of stars with great proper motion, it is more advantageous to start from the mean value of R. According to MALMQUIST we have the relation

$$\overline{R} = e^{-bM_0 + \frac{1}{2}b^2\sigma^2} \frac{A(m + b\sigma^2)}{A(m)}. \tag{36}$$

The mean value of R may be computed from

$$\overline{R} = M(R),$$

where R is computed from

$$R = \bar{s} \cdot \frac{1}{\mu} \cdot 10^{-0.2m}.$$

Using this formula I have determined the value of R for each star. This value turned out to be very great for certain stars, namely those with very small proper motions. For such stars the computation of R is very uncertain. I have therefore rejected seven stars before determining the mean value of R.[1])

The result is

$$\overline{R} = 3.184.$$

From (31) we get

$$\frac{A(m + b\sigma^2)}{A(m)} = e^{2.760\, b^2\sigma^2},$$

consequently

$$-5\,{}^{10}\log \overline{R} = M_0 - \frac{1}{2} b\sigma^2 - 2.760\, b\sigma^2.$$

Using the above value of $\overline{R}$ in this equation we get

$$-2.51 = M_0 - 1.502\,\sigma^2. \tag{37}$$

If in this equation we put $\sigma = 2.63$, we get

$$M_0 = +7.95.$$

If we combine the equation (34) with the equation (37) we get the following result:—

$$M_0 = +4.51,$$
$$\sigma = 2.16.$$

From fig. 4 it appears that the mean value of $\log s$ is smaller than the value of $\log s$, which corresponds to the mode of $\Phi_1(x)$. If the mean value of $\log s$ were used instead of the mode, $\overline{M}_{-\infty, m_1}$ would be somewhat greater. A greater value of $\overline{M}_{-\infty, m_1}$ would in its turn give greater values of M_0 and σ, if the equations (34) and (37) were solved together.

[1]) The rejection of these stars have no influence upon the coefficient of m in the formula (31).

B. STARS WITH KNOWN PARALLAX.

19. To test the reliability of the method of computing the absolute magnitude, I have applied it to stars of spectral type *G* from ADAMS and YOY's[1]) catalogue of stars with known parallax. In this catalogue the spectral types are given from Harvard 50 and, besides, the catalogue contains the spectral types determined in other ways. I have, however, accepted the spectral type from Harvard 50, as this is used in the other material in this paper, though ADAMS and YOY's method for determining the spectral type is probably more reliable.

The catalogue contains the total amount of the proper motion, but not the angle of position. This quantity I have taken partly from the BOSS catalogue and partly from VAN MAANEN:—"List of stars with proper motion exceeding $0''.50$ annually".[2])

In the same way and on the same assumption as before I have computed the absolute magnitude.[3]) The values thus obtained I have denoted M_c, and these values I have compared with the corresponding values M_A determined by ADAMS and YOY and then sought the correlation between M_c and M_A. I got the following result.

	The mean	The dispersion
M_A	$+ 2.^{m}486 \pm 0.^{m}189$	$2.^{m}234 \pm 0.^{m}133$,
M_c	$+ 2.^{m}857 \pm 0.^{m}329$	$3.^{m}890 \pm 0.^{m}232$.

The correlation coefficient is 0.855 ± 0.0227; the equations of regression are:—

$$M_A = 0.491\, M_c + 1.083,$$
$$M_c = 1.489\, M_A - 0.843.$$

We find that the mean of M_c is greater than that of M_A. The difference is, however, not great with regard to the mean errors. We further find that the dispersion of M_c is about twice the dispersion of M_A. This was to be expected, as the dispersion depends not only on the distribution of the absolute magnitude, but also on the uncertainty of the determination of the absolute magnitude; and the uncertainty in the determination of M_c is, of course, greater than in the determination of M_A. In fig. 6 I give the table of correlation with the lines of regression.

CORRELATION BETWEEN M_A AND M_c.

M_A \ M_c	−8 −6	−6 −4	−4 −2	−2 0	0 +2	+2 +4	+4 +6	+6 +8	+8 +10	+10 +12	
−8 −6											
−6 −4											
−4 −2	2										2
−2 0	5	4	7	9	3						28
0 +2				2	6	3	1				12
+2 +4					7	18	29	8			62
+4 +6						2	18	10	3	1	34
+6 +8							1	1			2
+8 +10											
+10 +12											
	7	4	7	11	16	23	49	19	3	1	

Fig. 6.

[1]) Ap. J. 46.

[2]) Ap. J. 41.

[3]) Strictly speaking, the absolute magnitude was computed by the formula $M = m - 5 \log \frac{l + \sqrt{l^2 + 4}}{2k} +$ $+ 5 \log \mu$ and then corrected with the help of the formula (26*). When I made these computations I had not thought of using the mode of $\Phi_1(x)$ instead of the mode of $\Phi(s)$. The figures in the first column of table VII have not been corrected in the above-mentioned way.

If there were a complete accordance between the two determinations of the absolute magnitude, the figures would lie accumulated along the diagonal. In any case, if the mean errors of the two determinations were the same, the diagonal would be the bisectrix of the angle formed by the lines of regression. If the uncertainty of the absolute magnitude should be greater in the one case than in the other, it might be expected that the one of the lines of regression would very nearly coicide with the diagonal. We find, however, that neither of these two cases occurs. The bisectrix does not coincide with the diagonal, but forms to this an angle of 16°.5. This indicates that there must be some systematical error in either method or, perhaps, in both.

Systematical errors in the parallax of stars (ADAMS and YOY's stars included) have been treated among others by BOSS.[1]) As, however, the discussion on the errors in the parallax of stars is still in progress and as these errors as regards ADAMS and YOY's stars cannot be so very great, we may, to begin with at least, assume ADAMS and YOY's determination of the absolute magnitude to be correct. We must then try to find an explanation of the peculiar phenomenon that, in the computation of the absolute magnitude by means of the method described above, the brighter stars appear too bright and the fainter stars too faint. As already mentioned a difference in the velocities of giants and of dwarfs would bring about systematical errors of this kind.

That the linear peculiar velocities of the stars increase with age is found by KAPTEYN.[2]) This conclusion has been confirmed later from more extensive data as to the radial velocities by CAMPHELL.[3])

20. To examine what effect the increasing velocities with decreasing brightness have upon the determination of the absolute magnitude, I have taken from GYLLENBERG's card-catalogue the stars brighter than $6^m.0$ with known radial velocities. To this material I have added the stars from ADAMS and YOY's catalogue with known radial velocities. Their radial velocities I have taken from WALTER S. ADAM's catalogue.[4])

The material thus obtained I have divided into groups according to the absolute magnitude. For each group I have computed the length λ of the velocity ellipsoid by the formula (29). It was necessary to reject one star on account of very extreme radial velocity. The result is shown in the table below. I have used the same value of α and ω as before.

TABLE VII.

Abs. magn.	n	λ	$\varepsilon(\lambda)$	λ_1	$\lambda - \lambda_1$
−8 −6	9	3.4	0.80	3.5	−0.1
−6 −4	34	3.7	0.45	3.5	+0.2
−4 −2	55	3.8	0.36	3.9	−0.1
−2 0	38	4.9	0.56	4.9	0.0
0 +2	20	6.8	1.07	6.3	+0.5
+2 +4	21	7.5	1.16	8.3	−0.8
+4 +6	22	11.2	1.69	10.7	+0.5
+6 +8	15	13.6	2.48	13.7	−0.1

[1]) "On the real motions of stars" A. J. 33, 771.
[2]) Ap. J. 31.
[3]) LICK, Observatory Bulletin, 196.
[4]) Ap. J. 42.

We find that the value of λ increases regularly with increasing absolute magnitude. The second column contains the number of stars in each group. The values $\varepsilon(\lambda)$ in the fourth column are computed from

$$\varepsilon(\lambda) = \frac{\lambda}{\sqrt{2n}}.$$

The values given in the fifth column are computed from the interpolation-formula

$$\lambda = 5.56 + 0.72\, M_c + 0.06\, M_c^2. \tag{38}$$

That the ellipsoid is greater for dwarfs than for giants was also found by EDDINGTON and HARTLEY.[1]) For F- and G-stars they have got the result

	Greatest axis	
Giants.....	16.95 km./sek.	3.58 sir./st.,
Dwarfs.....	28.44 «	6.22 « .

With the help of the formula (38) I have computed the value of λ for each star and then computed the absolute magnitude M_c' in the same way as before. The uncertainty of the computation of the absolute magnitude is very great for stars with small proper motions, as, for these stars, the uncertainty of the determination of the proper motion also affects the result. I have therefore rejected a few stars where the value of μ is smaller than $0''.015$.

I have now sought the correlation between M_c' and M_A. I get the following result:—

	The mean	The dispersion
M_A	$+2.684 \pm 0.182$	2.106 ± 0.129,
M_c'	$+2.384 \pm 0.226$	2.608 ± 0.160.

The correlation coefficient is now 0.830 ± 0.0269 and the lines of regression:—

$$M_A = 0.670\, M_c' + 1.086,$$
$$M_c' = 1.028\, M_A - 0.376.$$

We at once see that the dispersion of the computed absolute magnitude is smaller than before. The mean value of M_c' is now smaller than the mean value of M_A, but the difference is still small with regard to the mean error. The table of correlation is shown in fig. 7. One of the lines of regression very nearly coincides with the diagonal. This indicates a better accordance. Thus it proves necessary to take into consideration the size of the velocity ellipsoid for different absolute magnitudes. If, however, we examine table VII we find

CORRELATION BETWEEN M_A AND M_c'.

M_c' / M_A	−8 −6	−6 −4	−4 −2	−2 0	0 +2	+2 +4	+4 +6	+6 +8	+8 +10	+10 +12	
−8 −6											
−6 −4											
−4 −2		1									1
−2 0		1	9	10	2						22
0 +2				2	7	2	1				12
+2 +4					15	34	13				62
+4 +6						13	17	3	1		34
+6 +8							2				2
+8 +10											
+10 +12											
		2	9	12	24	49	33	3	1		

Fig. 7.

[1]) M. N. 75.

that if, for instance, the mean value of M is negative and the dispersion does not exceed $2^{m}.5$, the variation of the value of γ is not so great as to affect in any considerable measure the computation of M.

21. In the following table I have put together the stars rejected on account of small proper motions. The first column contains the numbers in ADAMS and YOY's catalogue.

No.	M_A	M_c'
6	—1.8	—3.5
47	—0.1	—6.9
84	—2.4	—5.8
119	—1.1	—5.5
139	—1.3	—4.9
405	—1.0	—5.3
455	—1.4	—6.4

They are exclusively bright stars. We see that for these stars M_c' has a considerably greater value than M_A, so that, if these stars were included in the correlation table, they would lie outside the lines of regression. As has already been mentioned, the mean error of μ considerably affects the computations of M_1, and it is therefore difficult to decide whether the difference is due to a mere chance or to other causes.

C. THE STARS OF THE GREENWICH CATALOGUE FOR 1910.0 PART II.

22. The proper motions of stars down to the ninth magnitude are recorded in the Greenwich catalogue for 1910 Part II. For a great many of these stars the spectral type has been determined from the HENRY DRAPER catalogue. The magnitude is referred to the Harvard scale. As the catalogue adopts only stars between $+24°$ and $+32°$ declination, a division into squares according to galactic coordinates is not suitable. I have therefore retained equatorial coordinates and divided the zone into 24 squares, so that the stars whose R. A. lies between 0^h and 1^h fall into the first square, those whose R. A. lies between 1^h and 2^h fall into the second, and so on. For these squares I have drawn up tables similar to those found in this investigation, and in so doing I have made the same supposition concerning the apex and the vertex as before. With the help of these tables I have then determined $\frac{1}{k}$ and l on the supposition that $\omega = 3.96$ sir./st. and $\lambda = 5.654$ sir./st.

In computing $\frac{1}{k}$ and l I have avoided interpolation and instead put $\theta = 0°$ for all stars whose value of θ lies between $-5°$ and $+5°$, $\theta = 10°$ for all stars whose value of θ lies between $+5°$ and $+15°$, and so on. I have then computed the absolute magnitude by means of the formula (26). The frequency distribution of M is to be found in table VIII.

TABLE VIII.

FREQUENCY DISTRIBUTION OF THE ABSOLUTE MAGNITUDE *M*.

M	$m < 8.55$					$m > 8.55$	All stars
	G_0	G_5	Gal. stars	Non-gal. stars	All stars < 8.55		
−8 −7						1	1
−7 −6		1	1		1		1
−6 −5						2	2
−5 −4	3	4	3	4	7	2	9
−4 −3	5	14	9	10	19	7	26
−3 −2	6	13	11	8	19	13	32
−2 −1	3	42	24	21	45	27	72
−1 0	24	82	60	46	106	65	171
0 +1	22	80	54	48	102	109	211
+1 +2	28	61	53	36	89	139	228
+2 +3	34	48	48	34	82	131	213
+3 +4	21	37	29	29	58	92	150
+4 +5	10	18	14	14	28	51	79
+5 +6	7	4	8	3	11	10	21
+6 +7	2		1	1	2	1	3
N	165	404	315	254	569	650	1219

FREQUENCY DISTRIBUTION OF THE ABSOLUTE MAGNITUDE M.
STARS BRIGHTER THAN 8m55 BETWEEN +24° AND +32° DECL.

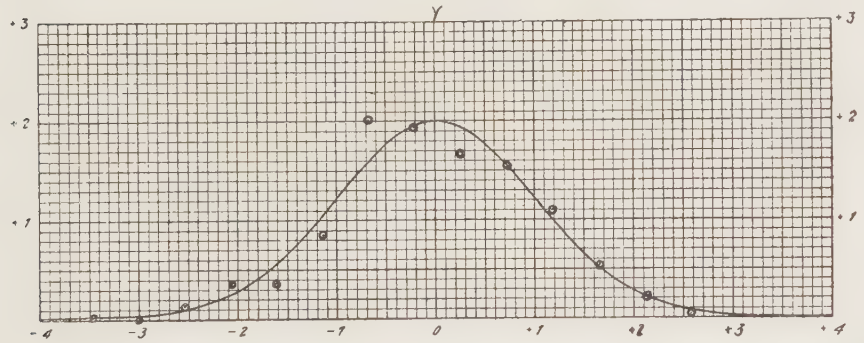

Fig. 8.

$$X = \frac{M - \overline{M}}{\sigma} \qquad Y = \frac{5\,\sigma}{N} F(M)$$

I have presumed the catalogue to be complete down to the magnitude 8.55, and therefore divided the material into stars brighter than 8.55 and stars fainter than 8.55. The former I have further divided into subclasses (G_0 and G_5) and into galactic and non-galactic stars. In the latter division all squares which lie approximately in the galactic zone (b between $\pm$ 30) have been taken together. The means and dispersions are to be found in table IX.

TABLE IX.

THE MEANS AND DISPERSIONS OF THE ABSOLUTE MAGNITUDE.

	Mean	Dispersion
All stars	$+1.33 \pm 0.059$	2.06 ± 0.042
$< 8^{m}.55$	$+0.96 \pm 0.090$	2.15 ± 0.064
$> 8^{m}.55$	$+1.66 \pm 0.075$	1.91 ± 0.053
$G_0 < 8^{m}.55$	$+1.54 \pm 0.181$	2.32 ± 0.127
$G_5 < 8^{m}.55$	$+0.71 \pm 0.102$	2.05 ± 0.072
Gal. stars $< 8^{m}.55$	$+0.99 \pm 0.118$	2.10 ± 0.084
Non-gal. stars $< 8^{m}.55$	$+0.93 \pm 0.136$	2.17 ± 0.096

23. We see here, as before, that the absolute magnitude increases with the apparent magnitude. The difference in absolute magnitude for G_0 and G_5 is obvious, whereas for galactic and non-galactic stars the means are about the same. It is worthy of notice that the dispersion for the stars of the Greenwich catalogue is considerably smaller than for the BOSS stars. This is certainly due to the fact that the number of giants does not increase in the same proportion as that of dwarfs, when apparently fainter stars are included. Stars of great absolute, but of small apparent brightness must lie so far off that the proper motions of these stars are very difficult to determine. And it is probable that in the determination of the proper motion of such a star the value obtained will be too big rather than too small, and a too great value of the proper motion gives a too small value of the brightness of the star, if this method of computing the absolute magnitude is used. This may also contribute to diminish the dispersion.

Another circumstance that highly influences the dispersion is the number of stars with very great proper motions. Very great proper motions are rare in the Greenwich catalogue, while the occurrence of great proper motions is characteristic of the apparently brighter G stars.

Assuming the uncertainty attaching to the computation of the absolute magnitude to be the same as in the case of the BOSS stars, we may use the same value of ε^2 as before. We thus get

$$\sigma = \sqrt{(2.15)^2 - 1.37} = 1.80.$$

TABLE X.

NUMBER OF STARS BRIGHTER THAN m.

m	$A(m)$ observed	$A(m)$ computed	Δ
6.05	28	24	+ 4
6.55	48	45	+ 3
7.05	92	85	+ 7
7.55	151	161	— 10
8.05	316	306	+ 10
8.55	570	578	— 8

24. The number of stars brighter than m is given in table X. The values of $A(m)$ in the third column are computed by means of the formula

$$^{10}\log A(m) = 0.555\, m - 1.983. \tag{39}$$

We see that $\log A(m)$ is fairly represented by a linear equation, and further we find that the coefficient of m is the same as the corresponding coefficient for the BOSS stars.

From the formula (39) we deduce

$$\frac{d \log A(m)}{d m} = 1.278.$$

Using the value of $\overline{M}_{-\infty,m}$ for the Greenwich stars brighter than $8^{m}.55$, we get the following relation

$$+ 0.96 = M_0 - 1.278\, \sigma^2. \tag{a}$$

25. As the dispersion of the absolute magnitude for the Greenwich stars is small and very great proper motions are rare, I have thought fit to apply the same method used in the first chapter for computing the mean value of $\frac{1}{R}$. But I have here made use of a simplification by taking the mean value of u' and v' for all stars in a square, and in this way I get 24 equations of the form

$$\overline{u'} = \gamma_{11}\left(\frac{\overline{U''}}{R}\right) + \gamma_{21}\left(\frac{\overline{V''}}{R}\right)$$

and 24 of the form

$$\overline{v'} = \gamma_{12}\left(\frac{\overline{U''}}{R}\right) + \gamma_{22}\left(\frac{\overline{V''}}{R}\right) + \gamma_{32}\left(\frac{\overline{W''}}{R}\right).$$

From these two groups of equations the mean value of $\frac{U''}{R}$, $\frac{V''}{R}$ and $\frac{W''}{R}$ was computed. In forming the normal equations, the corresponding equations from the motions in declination and in R. A. were added together. One star with extreme proper motion was excluded, likewise the two components of a tripple star.

The table XI contains the number of stars for different squares and the mean value of u' and v' for stars $< 8^{m}.55$.

The mean values of $\frac{U''}{R}$, $\frac{V''}{R}$ and $\frac{W''}{R}$ are found in table XII. The values of $\frac{1}{R}$ in the fifth column are computed on the supposition that 3.960 sir./st. is the velocity of the sun. The sixth column contains the values of M obtained from

$$M = 5\, ^{10}\log\left(\frac{1}{R}\right).$$

TABLE XI.

Square	Galactic latitude	Number of stars		$m < 8^{m}.55$	
		$< 8^{m}.55$	$\geq 8^{m}.55$	$\overline{u'}$	$\overline{v'}$
1	— 35	31	42	+ 1″.040	— 0″.715
2	— 33	19	32	+ 2″.433	— 0″.193
3	— 29	34	28	+ 0″.844	— 1″.494
4	— 21	22	21	+ 0″.757	— 2″.459
5	— 12	16	18	+ 0″.505	— 1″.707
6	— 1	24	15	— 0″.340	— 1″.083
7	+ 10	40	19	+ 0″.536	— 1″.147
8	+ 22	33	20	+ 0″.016	— 2″.866
9	+ 35	24	27	+ 0″.090	— 1″.680
10	+ 48	23	36	— 1″.664	— 1″.972
11	+ 61	24	46	+ 0″.240	— 3″.161
12	+ 75	13	34	— 1″.630	— 0″.251
13	+ 87	25	69	— 1″.801	— 1″.603
14	+ 79	25	35	— 1″.755	+ 0″.002
15	+ 66	22	21	— 0″.341	— 1″.095
16	+ 53	16	45	— 1″.198	— 1″.720
17	+ 40	19	27	— 1″.022	— 0″.394
18	+ 27	15	24	— 0″.357	+ 0″.836
19	+ 15	36	14	+ 1″.123	— 0″.416
20	+ 3	35	4	+ 1″.228	+ 0″.434
21	— 8	21	6	+ 1″.065	— 0″.518
22	— 18	22	20	+ 2″.084	— 0″.921
23	— 26	15	21	+ 1″.163	— 1″.207
24	— 32	14	25	+ 4″.018	— 1″.551

TABLE XII.

	Number of stars	$\left(\frac{\overline{U''}}{R}\right)$	$\left(\frac{\overline{V''}}{R}\right)$	$\left(\frac{\overline{W''}}{R}\right)$	$\left(\frac{\overline{1}}{R}\right)$	M	A	D	L	B
$< 8^{m}.55$	568	-0.333 ± 1.372	$+8.119 \pm 1.372$	-6.153 ± 1.214	2.574 ± 0.346	+2.05	272°.35	+37°.14	31°.45	+22°.15
$> 8^{m}.55$	649	-0.526 ± 1.718	$+6.158 \pm 1.718$	-7.463 ± 1.520	2.445 ± 0.415	+1.94	274°.89	+50°.09	45°.81	+24°.32
Abs. magn. > 0	903	-0.373 ± 1.504	$+8.690 \pm 1.504$	-8.481 ± 1.331	3.068 ± 0.374	+2.43	272°.46	+44°.27	38°.88	+24°.25
Abs. magn. < 0	314	-0.1690 ± 0.250	$+1.848 \pm 0.250$	-0.828 ± 0.221	0.513 ± 0.201	—1.45	275°.23	+24°.04	19°.60	+14°.98

The last columns contain the coordinates of the apex obtained from the relations

$$\left(\frac{\overline{U''}}{R}\right) = \omega \cos A \cos D,$$

$$\left(\frac{\overline{V''}}{R}\right) = \omega \sin A \cos D,$$

$$\left(\frac{\overline{W''}}{R}\right) = \omega \sin D.$$

26. We find that $\frac{\overline{1}}{R}$ has about the same value for stars brighter than $8^{m}.55$ and stars fainter than $8^{m}.55$. It might be expected that $\frac{\overline{1}}{R}$ should be greater for stars fainter than $8^{m}.55$ according to the table IX. It must be remembered, however, that we know nothing of how homogeneous the material is for which $m > 8^{m}.55$.

From the mean value of $\frac{1}{R}$ we get a new relation between M_0 and σ. From (39) we have

$$\frac{A(m - b\sigma^2)}{A(m)} = 10^{-0.555\, b\sigma^2}.$$

Putting this value in the formula (35) and taking the logarithms of both members, we get for stars brighter than $8^{m}.55$

(b) $$+ 2.05 = M_0 - 1.047\, \sigma^2.$$

27. A third relation between M_0 and σ is computed from the mean value of R. Using the formula

$$R = \bar{s} \cdot \frac{1}{\mu} 10^{-0.2\, m},$$

I have determined the value of R for each star brighter than $8^{m}.55$, and then computed the mean value. The result is ($n = 570$)

$$\overline{R} = 1.036.$$

Putting this value in (36), we get

(c) $$- 0.08 = M_0 - 1.508\, \sigma^2.$$

From the equations (a), (b) and (c) we get three different values of M_0 and σ respectively by combining them two and two. The result is

$$\text{(a, b)} \begin{cases} M_0 = +6.99, \\ \sigma = 2.17, \end{cases} \quad \text{(a, c)} \begin{cases} M_0 = +6.74, \\ \sigma = 2.12, \end{cases} \quad \text{(b, c)} \begin{cases} M_0 = +6.89, \\ \sigma = 2.15. \end{cases}$$

The accordance between the values of M_0 and σ thus obtained is, as we see, very good. And this result is also in fairly good accordance with the results from the first section of this chapter.

A problem that demands a further investigation is the velocity distribution of the G-stars and especially the relation of this distribution to the absolute magnitude. The provisional treatment of this problem in this chapter refers only to the length of the ellipsoid. It is probable that its form and position also have a relation to the absolute magnitude.

In concluding I desire to express my hearty thanks to my teacher, Professor C. V. L. CHARLIER, who has suggested to me to undertake this investigation and who has followed it with never-failing interest. I am also indebted to my friends Docent W. GYLLENBERG, Docent G. MALMQUIST and Amanuens OHLSSON for much valuable advice and suggestive discussion on astronomical problems.

THE PARAMETERS OF THE FUNCTION $\Phi(s) = \frac{2s}{\sqrt{2\pi}} e^{-\frac{(ks-l)^2}{2}}$

TABLE I. $L = \frac{\lambda}{\omega} \cdot l.$

$a = 0.5.$

θ	0	10	20	30	40	50	60	70	80	90	
GA_1	+ 1.2860	+ 1.3092	+ 1.3411	+ 1.3262	+ 1.2988	+ 1.2514	+ 1.2016	+ 1.1447	+ 1.0787	+ 1.0008	GF_2
GA_2	— 1.3408	— 1.4596	— 1.5246	— 1.5305	— 1.5124	— 1.4798	— 1.4369	— 1.3848	— 1.3217	— 1.2448	GF_1
GB_1	— 0.4768	— 0.3707	— 0.2257	— 0.0370	+ 0.1804	+ 0.3870	+ 0.5458	+ 0.6476	+ 0.7048	+ 0.7329	GE_6
GB_2	+ 0.3388	+ 0.5096	+ 0.6275	+ 0.6728	+ 0.6683	+ 0.6865	+ 0.5971	+ 0.5549	+ 0.5120	+ 0.4682	GE_7
GB_3	+ 0.9478	+ 0.9174	+ 0.8505	+ 0.7731	+ 0.6952	+ 0.6200	+ 0.5465	+ 0.4733	+ 0.3981	+ 0.3182	GE_8
GB_4	+ 1.1802	+ 1.0923	+ 0.9986	+ 0.9012	+ 0.7989	+ 0.6898	+ 0.5707	+ 0.4365	+ 0.2810	+ 0.0962	GE_9
GB_5	+ 1.2761	+ 1.1621	+ 1.0273	+ 0.8649	+ 0.6661	+ 0.4200	+ 0.1163	— 0.2446	— 0.6385	— 1.0101	GE_{10}
GB_6	+ 0.9157	+ 0.6734	+ 0.3495	— 0.0626	— 0.5257	— 0.9532	— 1.2695	— 1.4615	— 1.5598	— 1.5986	GE_1
GB_7	— 0.0583	— 0.5626	— 1.0160	— 1.3205	— 1.4817	— 1.5504	— 1.5679	— 1.5571	— 1.5299	— 1.4904	GE_2
GB_8	— 1.2817	— 1.4860	— 1.5769	— 1.6028	— 1.5934	— 1.5645	— 1.5224	— 1.4704	— 1.4071	— 1.3317	GE_3
GB_9	— 1.5959	— 1.6035	— 1.5852	— 1.5493	— 1.4982	— 1.4323	— 1.3495	— 1.2447	— 1.1104	— 0.9842	GE_4
GB_{10}	— 1.2454	— 1.2047	— 1.1460	— 1.0645	— 0.9526	— 0.7996	— 0.5920	— 0.3208	+ 0.0066	+ 0.3585	GE_5
GC_1	— 0.1432	— 0.1948	— 0.2377	— 0.2657	— 0.2768	— 0.2738	— 0.2612	— 0.2428	— 0.2209	— 0.1969	GD_7
GC_2	+ 0.4127	+ 0.3040	+ 0.1707	+ 0.0445	— 0.0590	— 0.1899	— 0.2033	— 0.2545	— 0.2972	— 0.3339	GD_8
GC_3	+ 0.7165	+ 0.5385	+ 0.3410	+ 0.1670	+ 0.0255	— 0.0880	— 0.1815	— 0.2615	— 0.3326	— 0.3982	GD_9
GC_4	+ 0.9555	+ 0.7557	+ 0.5472	+ 0.3562	+ 0.1893	+ 0.0437	— 0.0857	— 0.2040	— 0.3154	— 0.4235	GD_{10}
GC_5	+ 1.2485	+ 1.0730	+ 0.8797	+ 0.6794	+ 0.4778	+ 0.2769	+ 0.0766	— 0.1244	— 0.3271	— 0.5322	GD_{11}
GC_6	+ 1.2055	+ 1.0286	+ 0.8183	+ 0.5771	+ 0.3099	+ 0.0245	— 0.2686	— 0.5558	— 0.8229	— 1.0574	GD_{12}
GC_7	+ 0.4685	+ 0.1967	— 0.1036	— 0.3968	— 0.6494	— 0.8459	— 0.9879	— 1.0851	— 1.1475	— 1.1836	GD_1
GC_8	— 0.1976	— 0.5453	— 0.8246	— 1.0014	— 1.0939	— 1.1336	— 1.1428	— 1.1339	— 1.1128	— 1.0827	GD_2
GC_9	— 0.7629	— 1.0487	— 1.2007	— 1.4226	— 1.2531	— 1.2268	— 1.1863	— 1.1387	— 1.0849	— 1.0248	GD_3
GC_{10}	— 1.2331	— 1.3897	— 1.4446	— 1.4361	— 1.3928	— 1.3310	— 1.2583	— 1.1768	— 1.0864	— 0.9854	GD_4
GC_{11}	— 1.5417	— 1.6087	— 1.6203	— 1.5892	— 1.5249	— 1.4385	— 1.3183	— 1.1796	— 1.0162	— 0.8250	GD_5
GC_{12}	— 1.1280	— 1.1649	— 1.1705	— 1.1416	— 1.0758	— 0.9722	— 0.8321	— 0.6596	— 0.4631	— 0.2521	GD_6
	180	170	160	150	140	130	120	110	100	90	θ

$a = 0.5.$

θ	90	100	110	120	130	140	150	160	170	180	
GA_1	+ 1.0008	+ 0.9060	+ 0.7857	+ 0.6269	+ 0.4107	+ 0.1140	— 0.2706	— 0.6907	— 1.0339	— 1.2360	GF_2
GA_2	— 1.2448	— 1.1488	— 1.0241	— 0.8556	— 0.6211	— 0.2914	+ 0.1482	+ 0.6454	+ 1.0711	+ 1.3408	GF_1
GB_1	+ 0.7329	+ 0.7426	+ 0.7403	+ 0.7297	+ 0.7118	+ 0.6870	+ 0.6543	+ 0.6113	+ 0.5541	+ 0.4768	GE_6
GB_2	+ 0.4682	+ 0.4229	+ 0.3746	+ 0.3210	+ 0.2597	+ 0.1867	+ 0.0963	— 0.0188	— 0.1650	— 0.3388	GE_7
GB_3	+ 0.3182	+ 0.2298	+ 0.1285	+ 0.0079	— 0.1396	— 0.3212	— 0.5345	— 0.7481	— 0.8990	— 0.9478	GE_8
GB_4	+ 0.0962	— 0.1267	— 0.3922	— 0.6892	— 0.9757	— 1.1890	— 1.2929	— 1.3030	— 1.2557	— 1.1802	GE_9
GB_5	— 1.0101	— 1.2998	— 1.4820	— 1.5700	— 1.5920	— 1.5709	— 1.5225	— 1.4555	— 1.3731	— 1.2761	GE_{10}
GB_6	— 1.5986	— 1.6007	— 1.5792	— 1.5414	— 1.4890	— 1.4225	— 1.3388	— 1.2330	— 1.0960	— 0.9157	GE_1
GB_7	— 1.4904	— 1.4413	— 1.3807	— 1.3060	— 1.2121	— 1.0904	— 0.9261	— 0.6974	— 0.3752	+ 0.0583	GE_2
GB_8	— 1.3317	— 1.2384	— 1.1204	— 0.9653	— 0.7544	— 0.4618	— 0.0623	+ 0.4306	+ 0.9195	+ 1.2817	GE_3
GB_9	— 0.9842	— 0.6991	— 0.3865	+ 0.0115	+ 0.4654	+ 0.8969	+ 1.2281	+ 1.4373	+ 1.5488	+ 1.5959	GE_4
GB_{10}	+ 0.3585	+ 0.6666	— 0.9090	+ 1.0757	+ 1.1810	+ 1.2420	+ 1.2720	+ 1.2800	+ 1.2704	+ 1.2454	GE_5
GC_1	— 0.1969	— 0.1712	— 0.1437	— 0.1140	— 0.0817	— 0.0458	— 0.0056	+ 0.0397	+ 0.0900	+ 0.1432	GD_7
GC_2	— 0.3339	— 0.3667	— 0.3967	— 0.4245	— 0.4503	— 0.4732	— 0.4904	— 0.4950	— 0.4747	— 0.4127	GD_8
GC_3	— 0.3982	— 0.4611	— 0.5235	— 0.5871	— 0.6528	— 0.7200	— 0.7834	— 0.8267	— 0.8162	— 0.7165	GD_9
GC_4	— 0.4235	— 0.5317	— 0.6425	— 0.7578	— 0.8770	— 0.9948	— 1.0938	— 1.1410	— 1.0972	— 0.9555	GD_{10}
GC_5	— 0.5322	— 0.7389	— 0.9435	— 1.1376	— 1.3068	— 1.4310	— 1.4910	— 1.4759	— 1.3894	— 1.2485	GD_{11}
GC_6	— 1.0574	— 1.2501	— 1.3963	— 1.4951	— 1.5476	— 1.5567	— 1.5249	— 1.4547	— 1.3477	— 1.2055	GD_{12}
GC_7	— 1.1836	— 1.1988	— 1.1956	— 1.1750	— 1.1357	— 1.0747	— 0.9858	— 0.8609	— 0.6909	— 0.4685	GD_1
GC_8	— 1.0827	— 1.0438	— 0.9949	— 0.9332	— 0.8541	— 0.7493	— 0.6064	— 0.4082	— 0.1384	+ 0.1976	GD_2
GC_9	— 1.0248	— 0.9563	— 0.8761	— 0.7788	— 0.6561	— 0.4952	— 0.2776	+ 0.0161	+ 0.3841	+ 0.7629	GD_3
GC_{10}	— 0.9854	— 0.8697	— 0.7833	— 0.5678	— 0.3618	— 0.1022	+ 0.2198	+ 0.5906	+ 0.9544	+ 1.2331	GD_4
GC_{11}	— 0.8250	— 0.6024	— 0.3450	— 0.0525	+ 0.2689	+ 0.6028	+ 0.9228	+ 1.1984	+ 1.4070	+ 1.5417	GD_5
GC_{12}	— 0.2521	— 0.0369	+ 0.1787	+ 0.3717	+ 0.5531	+ 0.7145	+ 0.8542	+ 0.9706	+ 1.0623	+ 1.1280	GD_6
	90	80	70	60	50	40	30	20	10	0	θ

$\alpha = 0.6$.

θ	0	10	20	30	40	50	60	70	80	90	
GA_1	+ 1.0908	+ 1.1648	+ 1.1831	+ 1.1673	+ 1.1310	+ 1.0820	+ 1.0229	+ 0.9545	+ 0.8757	+ 0.7834	GF_2
GA_2	— 1.1564	— 1.2715	— 1.3202	— 1.3261	— 1.3056	— 1.2681	— 1.2172	— 1.1546	— 1.0792	— 0.9882	GF_1
GB_1	— 0.3411	— 0.2408	— 0.1157	+ 0.0317	+ 0.1894	+ 0.3370	+ 0.4562	+ 0.5401	+ 0.5920	+ 0.6197	GE_6
GB_2	+ 0.3931	+ 0.5150	+ 0.5979	+ 0.6319	+ 0.6268	+ 0.5973	+ 0.5562	+ 0.5097	+ 0.4604	+ 0.4089	GE_7
GB_3	+ 0.9146	+ 0.8896	+ 0.8283	+ 0.7507	+ 0.6666	+ 0.5816	+ 0.4961	+ 0.4093	+ 0.3197	+ 0.2248	GE_8
GB_4	+ 1.0960	+ 1.0046	+ 0.9029	+ 0.7937	+ 0.6775	+ 0.5532	+ 0.4182	+ 0.2692	+ 0.1022	— 0.0864	GE_9
GB_5	+ 1.0557	+ 0.9304	+ 0.7845	+ 0.6137	+ 0.4138	+ 0.1806	— 0.0863	— 0.3787	— 0.6756	— 0.9457	GE_{10}
GB_6	+ 0.6252	+ 0.3961	+ 0.1166	— 0.2054	— 0.5414	— 0.8466	— 1.0849	— 1.2422	— 1.3313	— 1.3694	GE_1
GB_7	— 0.2234	— 0.5735	— 0.9022	— 1.1326	— 1.2693	— 1.3344	— 1.3523	— 1.3406	— 1.3090	— 1.2629	GE_2
GB_8	— 1.1183	— 1.2833	— 1.3655	— 1.3912	— 1.3809	— 1.3480	— 1.2993	— 1.2373	— 1.1623	— 1.0724	GE_3
GB_9	— 1.3659	— 1.3733	— 1.3587	— 1.3132	— 1.2552	— 1.1802	— 1.0863	— 0.9702	— 0.8258	— 0.6460	GE_4
GB_{10}	— 1.0356	— 0.9909	— 0.9274	— 0.8417	— 0.7292	— 0.5843	— 0.4019	— 0.1824	+ 0.0643	+ 0.3164	GE_5
GC_1	— 0.1516	— 0.1914	— 0.2283	— 0.2440	— 0.2526	— 0.2501	— 0.2390	— 0.2218	— 0.2002	— 0.1756	GD_7
GC_2	+ 0.3892	+ 0.3126	+ 0.2157	+ 0.1154	+ 0.0285	— 0.0556	— 0.1224	— 0.1790	— 0.2278	— 0.2707	GD_8
GC_3	+ 0.7216	+ 0.5945	+ 0.4397	+ 0.2856	+ 0.1468	+ 0.0261	— 0.0788	— 0.1719	— 0.2562	— 0.3348	GD_9
GC_4	+ 0.9717	+ 0.8183	+ 0.6421	+ 0.4647	+ 0.2972	+ 0.1424	— 0.0008	— 0.1349	— 0.2626	— 0.3867	GD_{10}
GC_5	+ 1.1681	+ 1.0191	+ 0.8470	+ 0.6608	+ 0.4670	+ 0.2693	+ 0.0695	— 0.1312	— 0.3317	— 0.5808	GD_{11}
GC_6	+ 0.9903	+ 0.8342	+ 0.6515	+ 0.4458	+ 0.2221	— 0.0127	— 0.2505	— 0.4816	— 0.6965	— 0.8863	GD_{12}
GC_7	+ 0.3872	+ 0.1280	— 0.0951	— 0.3133	— 0.5085	— 0.6698	— 0.7947	— 0.8854	— 0.9468	— 0.9838	GD_1
GC_8	— 0.2624	— 0.5076	— 0.7100	— 0.8501	— 0.9320	— 0.9707	— 0.9801	— 0.9701	— 0.9460	— 0.9108	GD_2
GC_9	— 0.7491	— 0.9522	— 1.0704	— 1.1169	— 1.1159	— 1.0864	— 1.0422	— 0.9869	— 0.9230	— 0.8510	GD_3
GC_{10}	— 1.1847	— 1.2542	— 1.2999	— 1.2916	— 1.2480	— 1.1822	— 1.1016	— 1.0094	— 0.9057	— 0.7898	GD_4
GC_{11}	— 1.3310	— 1.3875	— 1.3979	— 1.3689	— 1.3069	— 1.2170	— 1.1021	— 0.9637	— 0.8017	— 0.6164	GD_5
GC_{12}	— 0.9492	— 0.9818	— 0.9866	— 0.9621	— 0.9071	— 0.8219	— 0.7081	— 0.5694	— 0.4112	— 0.2404	GD_6
	180	170	160	150	140	130	120	110	100	90	θ

$\alpha = 0.6$.

θ	90	100	110	120	130	140	150	160	170	180	
GA_1	+ 0.7834	+ 0.6734	+ 0.5389	+ 0.3712	+ 0.1606	— 0.0987	— 0.3965	— 0.6941	— 0.9362	— 1.0908	GF_2
GA_2	— 0.9882	— 0.8767	— 0.7373	— 0.5594	— 0.3310	— 0.0429	+ 0.2974	+ 0.6495	+ 0.9487	+ 1.1564	GF_1
GB_1	+ 0.6197	+ 0.6297	+ 0.6273	+ 0.6152	+ 0.5947	+ 0.5663	+ 0.5290	+ 0.4812	+ 0.4198	+ 0.3411	GE_6
GB_2	+ 0.4089	+ 0.3549	+ 0.2971	+ 0.2338	+ 0.1628	+ 0.0811	— 0.0146	— 0.1275	— 0.2567	— 0.3931	GE_7
GB_3	+ 0.2248	+ 0.1217	+ 0.0068	— 0.1238	— 0.2732	— 0.4408	— 0.6162	— 0.7740	— 0.8796	— 0.9146	GE_8
GB_4	— 0.0864	— 0.2984	— 0.5298	— 0.7650	— 0.9752	— 1.1274	— 1.2044	— 1.2120	— 1.1696	— 1.0960	GE_9
GB_5	— 0.9457	— 1.1592	— 1.3013	— 1.3757	— 1.3952	— 1.3744	— 1.3245	— 1.2531	— 1.1628	— 1.0557	GE_{10}
GB_6	— 1.3694	— 1.3713	— 1.3479	— 1.3051	— 1.2453	— 1.1688	— 1.0736	— 0.9556	— 0.8088	— 0.6252	GE_1
GB_7	— 1.2629	— 1.2039	— 1.1316	— 1.0434	— 0.9349	— 0.7987	— 0.6248	— 0.4006	— 0.1164	+ 0.2234	GE_2
GB_8	— 1.0724	— 0.9638	— 0.8299	— 0.6619	— 0.4486	— 0.1788	+ 0.1495	+ 0.5125	+ 0.8547	+ 1.1183	GE_3
GB_9	— 0.6460	— 0.4225	— 0.1504	+ 0.1643	+ 0.4967	+ 0.8050	+ 1.0514	+ 1.2207	+ 1.3202	+ 1.3659	GE_4
GB_{10}	+ 0.3164	+ 0.5481	+ 0.7367	+ 0.8779	+ 0.9743	+ 1.0340	+ 1.0652	+ 1.0739	+ 1.0632	+ 1.0356	GE_5
GC_1	— 0.1756	— 0.1487	— 0.1195	— 0.0882	— 0.0544	+ 0.0177	+ 0.0218	+ 0.0640	+ 0.1081	+ 0.1516	GD_7
GC_2	— 0.2707	— 0.3091	— 0.3442	— 0.3764	— 0.4053	— 0.4299	— 0.4473	— 0.4517	— 0.4351	— 0.3892	GD_8
GC_3	— 0.3348	— 0.4101	— 0.4838	— 0.5570	— 0.6301	— 0.7003	— 0.7614	— 0.7990	— 0.7916	— 0.7216	GD_9
GC_4	— 0.3867	— 0.5092	— 0.6319	— 0.7546	— 0.8746	— 0.9842	— 1.0683	— 1.1049	— 1.0787	— 0.9717	GD_{10}
GC_5	— 0.5308	— 0.7253	— 0.9105	— 1.0782	— 1.2171	— 1.3152	— 1.3612	— 1.3495	— 1.2821	— 1.1681	GD_{11}
GC_6	— 0.8863	— 1.0445	— 1.1667	— 1.2505	— 1.2962	— 1.3040	— 1.2756	— 1.2128	— 1.1171	— 0.9903	GD_{12}
GC_7	— 0.9838	— 0.9994	— 0.9961	— 0.9745	— 0.9336	— 0.8713	— 0.7840	— 0.6677	— 0.5189	— 0.3372	GD_1
GC_8	— 0.9108	— 0.8650	— 0.8079	— 0.7369	— 0.6481	— 0.5357	— 0.3923	— 0.2103	+ 0.0122	+ 0.2624	GD_2
GC_9	— 0.8510	— 0.7692	— 0.6743	— 0.5621	— 0.4261	— 0.2580	— 0.0493	+ 0.2026	+ 0.4826	+ 0.7491	GD_3
GC_{10}	— 0.7898	— 0.6585	— 0.5075	— 0.3316	— 0.1248	+ 0.1164	+ 0.3885	+ 0.6731	+ 0.9347	+ 1.1847	GD_4
GC_{11}	— 0.6164	— 0.4069	— 0.1741	+ 0.0781	+ 0.3422	+ 0.6051	+ 0.8501	+ 1.0602	+ 1.2222	+ 1.3310	GD_5
GC_{12}	— 0.2404	— 0.0637	+ 0.1120	+ 0.2807	+ 0.4379	+ 0.5799	+ 0.7041	+ 0.8085	+ 0.8907	+ 0.9492	GD_6
	90	80	70	60	50	40	30	20	10	0	θ

$\alpha = 0.7$.

θ	0	10	20	30	40	50	60	70	80	90	
GA_1	+ 0.9965	+ 1.0596	+ 1.0764	+ 1.0602	+ 1.0211	+ 0.9662	+ 0.8983	+ 0.8187	+ 0.7269	+ 0.6207	GF_2
GA_2	— 1.0362	— 1.1345	— 1.1798	— 1.1857	— 1.1637	— 1.1215	— 1.0632	— 0.9902	— 0.9025	— 0.7978	GF_1
GB_1	— 0.2460	— 0.1533	— 0.0459	— 0.0722	+ 0.1925	+ 0.3040	+ 0.3969	+ 0.4666	+ 0.5130	+ 0.5395	GE_6
GB_2	+ 0.4264	+ 0.5177	+ 0.5790	+ 0.6054	+ 0.6004	+ 0.5734	+ 0.5323	+ 0.4828	+ 0.4280	+ 0.3692	GE_7
GB_3	+ 0.8925	+ 0.8718	+ 0.8169	+ 0.7413	+ 0.6544	+ 0.5616	+ 0.4652	+ 0.3655	+ 0.2619	+ 0.1527	GE_8
GB_4	+ 1.0354	+ 0.9437	+ 0.8869	+ 0.7190	+ 0.5914	+ 0.4543	+ 0.3065	+ 0.1467	— 0.0264	— 0.2129	GE_9
GB_5	+ 0.8983	+ 0.7653	+ 0.6128	+ 0.4393	+ 0.2489	+ 0.0278	— 0.2066	— 0.4482	— 0.6824	— 0.8905	GE_{10}
GB_6	+ 0.4219	+ 0.2103	— 0.0298	— 0.2877	— 0.5441	— 0.7748	— 0.9600	— 1.0913	— 1.1708	— 1.2071	GE_1
GB_7	— 0.3259	— 0.5954	— 0.8818	— 1.0107	— 1.1259	— 1.1858	— 1.2036	— 1.1910	— 1.1559	— 1.1031	GE_2
GB_8	— 1.0138	— 1.1480	— 1.2209	— 1.2454	— 1.2348	— 1.1989	— 1.1439	— 1.0726	— 0.9858	— 0.8823	GE_3
GB_9	— 1.2028	— 1.2102	— 1.1894	— 1.1459	— 1.0821	— 0.9992	— 0.8989	— 0.7720	— 0.6224	— 0.4450	GE_4
GB_{10}	— 0.8858	— 0.8383	— 0.7719	— 0.6849	— 0.5749	— 0.4403	— 0.2807	— 0.0993	+ 0.0951	+ 0.2894	GE_5
GC_1	— 0.1521	— 0.1841	— 0.2092	— 0.2255	— 0.2324	— 0.2302	— 0.2204	— 0.2045	— 0.1937	— 0.1593	GD_7
GC_2	+ 0.3728	+ 0.3156	+ 0.2423	+ 0.1612	+ 0.0805	+ 0.0056	— 0.0621	— 0.1225	— 0.1764	— 0.2247	GD_8
GC_3	+ 0.7246	+ 0.6297	+ 0.5063	+ 0.3723	+ 0.2404	+ 0.1169	+ 0.0033	— 0.1013	— 0.1984	— 0.2900	GD_9
GC_4	+ 0.9703	+ 0.8490	+ 0.6998	+ 0.5384	+ 0.3753	+ 0.2164	+ 0.0636	— 0.0833	— 0.2249	— 0.3624	GD_{10}
GC_5	+ 1.0848	+ 0.9563	+ 0.8028	+ 0.6314	+ 0.4484	+ 0.2582	+ 0.0641	— 0.1313	— 0.3252	— 0.5147	GD_{11}
GC_6	+ 0.8402	+ 0.7017	+ 0.5410	+ 0.3622	+ 0.1701	— 0.0294	— 0.2296	— 0.4233	— 0.6032	— 0.7628	GD_{12}
GC_7	+ 0.2554	+ 0.0874	— 0.0880	— 0.2596	— 0.4168	— 0.5521	— 0.6618	— 0.7457	— 0.8046	— 0.8411	GD_1
GC_8	— 0.2992	— 0.4820	— 0.6355	— 0.7483	— 0.8199	— 0.8563	— 0.8659	— 0.8551	— 0.8285	— 0.7887	GD_2
GC_9	— 0.7406	— 0.8921	— 0.9857	— 1.0256	— 1.0246	— 0.9951	— 0.9464	— 0.8839	— 0.8104	— 0.7266	GD_3
GC_{10}	— 1.0641	— 1.1582	— 1.1965	— 1.1887	— 1.1462	— 1.0785	— 0.9924	— 0.8913	— 0.7764	— 0.6479	GD_4
GC_{11}	— 1.1736	— 1.2223	— 1.2318	— 1.2044	— 1.1460	— 1.0594	— 0.9477	— 0.8129	— 0.6565	— 0.4800	GD_5
GC_{12}	— 0.8185	— 0.8477	— 0.8519	— 0.8306	— 0.7835	— 0.7111	— 0.6153	— 0.4991	— 0.3666	— 0.2229	GD_6
	180	170	160	150	140	130	120	110	100	90	θ

$\alpha = 0.7$.

θ	90	100	110	120	130	140	150	160	170	180	
GA_1	+ 0.6207	+ 0.4974	+ 0.3525	+ 0.1821	— 0.0164	— 0.2399	— 0.4747	— 0.6960	— 0.8756	— 0.9965	GF_2
GA_2	— 0.7978	— 0.6728	— 0.5226	— 0.3418	— 0.1265	+ 0.1215	+ 0.3899	+ 0.6517	+ 0.8744	+ 1.0362	GF_1
GB_1	+ 0.5395	+ 0.5497	+ 0.5470	+ 0.5338	+ 0.5111	+ 0.4796	+ 0.4387	+ 0.3871	+ 0.3235	+ 0.2460	GE_6
GB_2	+ 0.3692	+ 0.3068	+ 0.2398	+ 0.1672	+ 0.0877	— 0.0005	— 0.0982	— 0.2053	— 0.3176	— 0.4264	GE_7
GB_3	+ 0.1527	— 0.0363	— 0.0896	— 0.2258	— 0.3718	— 0.5226	— 0.6673	— 0.7879	— 0.8661	— 0.8925	GE_8
GB_4	— 0.2129	— 0.4098	— 0.6100	— 0.8005	— 0.9621	— 1.0774	— 1.1370	— 1.1430	— 1.1050	— 1.0354	GE_9
GB_5	— 0.8905	— 1.0564	— 1.1710	— 1.2342	— 1.2517	— 1.2316	— 1.1816	— 1.1078	— 1.0128	— 0.8983	GE_{10}
GB_6	— 1.2071	— 1.2091	— 1.1842	— 1.1373	— 1.0712	— 0.9863	— 0.8817	— 1.7550	— 0.6028	— 0.4219	GE_1
GB_7	— 1.1031	— 1.0350	— 0.9513	— 0.8500	— 0.7284	— 0.5816	— 0.4040	— 0.1915	+ 0.0555	+ 0.3259	GE_2
GB_8	— 0.8823	— 0.7596	— 0.6131	— 0.4181	— 0.2298	+ 0.0130	+ 0.2882	+ 0.5606	+ 0.8138	+ 1.0138	GE_3
GB_9	— 0.4450	— 0.2374	— 0.0018	— 0.2527	+ 0.5082	+ 0.7416	+ 0.9323	+ 1.0721	+ 1.1596	+ 1.2028	GE_4
GB_{10}	+ 0.2894	+ 0.4688	+ 0.6218	+ 0.7421	+ 0.8294	+ 0.8864	+ 0.9174	+ 0.9261	+ 0.9151	+ 0.8858	GE_5
GC_1	— 0.1593	— 0.1321	— 0.1022	— 0.0701	— 0.0359	+ 0.0004	+ 0.0382	+ 0.0771	+ 0.1156	+ 0.1521	GD_7
GC_2	— 0.2247	— 0.2685	— 0.3082	— 0.3441	— 0.3753	— 0.4008	— 0.4177	— 0.4218	— 0.4082	— 0.3728	GD_8
GC_3	— 0.2900	— 0.3775	— 0.4619	— 0.5438	— 0.6218	— 0.6926	— 0.7496	— 0.7818	— 0.7763	— 0.7246	GD_9
GC_4	— 0.3624	— 0.4968	— 0.6278	— 0.7589	— 0.8708	— 0.9708	— 1.0420	— 1.0713	— 1.0477	— 0.9703	GD_{10}
GC_5	— 0.5147	— 0.6957	— 0.8627	— 1.0091	— 1.1267	— 1.2072	— 1.2443	— 1.2349	— 1.1802	— 1.0848	GD_{11}
GC_6	— 0.7628	— 0.8970	— 1.0018	— 1.0746	— 1.1145	— 1.1214	— 1.0960	— 1.0396	— 0.9585	— 0.8402	GD_{12}
GC_7	— 0.8411	— 0.8571	— 0.8537	— 0.8315	— 0.7900	— 0.7288	— 0.6447	— 0.5378	— 0.4074	— 0.2554	GD_1
GC_8	— 0.7887	— 0.7367	— 0.6720	— 0.5929	— 0.4969	— 0.3810	— 0.2418	— 0.0781	— 0.1061	+ 0.2992	GD_2
GC_9	— 0.7266	— 0.6314	— 0.5227	— 0.3975	— 0.2520	— 0.0880	+ 0.1110	+ 0.3250	+ 0.5433	+ 0.7406	GD_3
GC_{10}	— 0.6479	— 0.5041	— 0.3429	— 0.1622	+ 0.0390	+ 0.2586	+ 0.4887	+ 0.7140	+ 0.9124	+ 1.0641	GD_4
GC_{11}	— 0.4800	— 0.2853	— 0.0753	+ 0.1449	+ 0.3679	+ 0.5838	+ 0.7816	+ 0.9507	+ 1.0830	+ 1.1736	GD_5
GC_{12}	— 0.2223	— 0.0730	+ 0.0775	+ 0.2240	+ 0.3619	+ 0.4880	+ 0.5990	+ 0.6925	+ 0.7663	+ 0.8185	GD_6
	90	80	70	60	50	40	30	20	10	0	θ

TABLE II. $K = \frac{1}{\lambda k}$.

$a = 0.5$.

θ	0	10	20	30	40	50	60	70	80	90	
$GA_1\ GA_2$	0.8627	0.7508	0.6702	0.6034	0.5558	0.5242	0.5061	0.5001	0.5055	0.5228	$GF_1\ GF_2$
$GB_1\ GB_6$	0.6300	0.6953	0.7699	0.8385	0.8751	0.8601	0.8022	0.7276	0.6571	0.5995	$GE_1\ GE_6$
$GB_2\ GB_7$	0.9337	0.9846	0.9558	0.8674	0.7655	0.6771	0.6090	0.5599	0.5270	0.5075	$GE_2\ GE_7$
$GB_3\ GB_8$	0.9018	0.8027	0.7087	0.6332	0.5772	0.5384	0.5138	0.5018	0.5011	0.5120	$GE_3\ GE_8$
$GB_4\ GB_9$	0.6044	0.5602	0.5289	0.5095	0.5007	0.5021	0.5138	0.5364	0.5712	0.6193	$GE_4\ GE_9$
$GB_5\ GB_{10}$	0.5003	0.5075	0.5242	0.5510	0.5883	0.6359	0.6905	0.7442	0.7826	0.7903	$GE_5\ GE_{10}$
$GC_1\ GC_7$	0.6849	0.7156	0.7267	0.7140	0.6822	0.6413	0.6001	0.5641	0.5356	0.5155	$GD_1\ GD_7$
$GC_2\ GC_8$	0.9197	0.9310	0.8783	0.7924	0.7058	0.6337	0.5790	0.5403	0.5153	0.5024	$GD_2\ GD_8$
$GC_3\ GC_9$	0.9966	0.9451	0.8455	0.7430	0.6583	0.5948	0.5499	0.5206	0.5044	0.5001	$GD_3\ GD_9$
$GC_4\ GC_{10}$	0.8508	0.7869	0.7119	0.6438	0.5891	0.5486	0.5212	0.5053	0.5000	0.5051	$GD_4\ GD_{10}$
$GC_5\ GC_{11}$	0.6050	0.5781	0.5525	0.5309	0.5146	0.5042	0.5000	0.5020	0.5102	0.5244	$GD_5\ GD_{11}$
$GC_6\ GC_{12}$	0.5048	0.5107	0.5183	0.5268	0.5353	0.5427	0.5478	0.5497	0.5483	0.5436	$GD_6\ GD_{12}$
	180	170	160	150	140	130	120	110	100	90	θ

$a = 0.5$.

θ	90	100	110	120	130	140	150	160	170	180	
$GA_1\ GA_2$	0.5228	0.5536	0.6003	0.6659	0.7529	0.8567	0.9544	0.9987	0.9587	0.8627	$GF_1\ GF_2$
$GB_1\ GB_6$	0.5995	0.5561	0.5259	0.5077	0.5002	0.5031	0.5165	0.5413	0.5786	0.6300	$GE_1\ GE_6$
$GB_2\ GB_7$	0.5075	0.5002	0.5042	0.5199	0.5486	0.5926	0.6548	0.7374	0.8369	0.9337	$GE_2\ GE_7$
$GB_3\ GB_8$	0.5120	0.5350	0.5722	0.6261	0.6994	0.7918	0.8916	0.9652	0.9697	0.9018	$GE_3\ GE_8$
$GB_4\ GB_9$	0.6193	0.6807	0.7515	0.8187	0.8588	0.8518	0.8014	0.7313	0.6623	0.6044	$GE_4\ GE_9$
$GB_5\ GB_{10}$	0.7903	0.7639	0.7148	0.6591	0.6079	0.5660	0.5346	0.5136	0.5023	0.5003	$GE_5\ GE_{10}$
$GC_1\ GC_7$	0.5155	0.5037	0.4999	0.5042	0.5166	0.5374	0.5664	0.6028	0.6443	0.6849	$GD_1\ GD_7$
$GC_2\ GC_8$	0.5024	0.5006	0.5098	0.5306	0.5646	0.6138	0.6802	0.7628	0.8518	0.9198	$GD_2\ GD_8$
$GC_3\ GC_9$	0.5001	0.5073	0.5266	0.5596	0.6089	0.6776	8.7674	0.8721	0.9646	0.9966	$GD_3\ GD_9$
$GC_4\ GC_{10}$	0.5051	0.5209	0.5482	0.5885	0.6429	0.7109	0.7859	0.8501	0.8766	0.8508	$GD_4\ GD_{10}$
$GC_5\ GC_{11}$	0.5244	0.5442	0.5686	0.5954	0.6211	0.6408	0.6498	0.6455	0.6291	0.6050	$GD_5\ GD_{11}$
$GC_6\ GC_{12}$	0.5436	0.5365	0.5280	0.5194	0.5116	0.5055	0.5015	0.5000	0.5011	0.5048	$GD_6\ DG_{12}$
	90	80	70	60	50	40	30	20	10	0	θ

$a = 0.6$.

θ	0	10	20	30	40	50	60	70	80	90	
$GA_1\ GA_2$	0.9112	0.8337	0.7609	0.7009	0.6556	0.6245	0.6062	0.6000	0.6056	0.6231	$GF_1\ GF_6$
$GB_1\ GB_6$	0.7184	0.7712	0.8261	0.8720	0.8948	0.8856	0.8482	0.7956	0.7408	0.6922	$GE_1\ GE_6$
$GB_2\ GB_7$	0.9560	0.9872	0.9698	0.9123	0.8379	0.7660	0.7056	0.6595	0.6272	0.6077	$GE_2\ GE_7$
$GB_3\ GB_8$	0.9336	0.8649	0.7917	0.7271	0.6757	0.6383	0.6140	0.6018	0.6012	0.6121	$GE_3\ GE_8$
$GB_4\ GB_9$	0.6958	0.6566	0.6277	0.6091	0.6006	0.6020	0.6133	0.6348	0.6666	0.7084	$GE_4\ GE_9$
$GB_5\ GB_{10}$	0.6003	0.6070	0.6224	0.6464	0.6784	0.7168	0.7578	0.7951	0.8197	0.8246	$GE_5\ GE_{10}$
$GC_1\ GC_7$	0.7455	0.7658	0.7729	0.7648	0.7438	0.7150	0.6841	0.6553	0.6314	0.6139	$GD_1\ GD_7$
$GC_2\ GC_8$	0.9356	0.9423	0.9095	0.8510	0.7853	0.7252	0.6761	0.6396	0.6152	0.6024	$GD_2\ GD_8$
$GC_3\ GC_9$	0.9974	0.9658	0.8989	0.8215	0.7505	0.6924	0.6499	0.6209	0.6045	0.6001	$GD_3\ GD_9$
$GC_4\ GC_{10}$	0.8798	0.8379	0.7838	0.7298	0.6830	0.6463	0.6204	0.6051	0.5999	0.6049	$GD_4\ GD_{10}$
$GC_5\ GC_{11}$	0.6818	0.6623	0.6429	0.6258	0.6124	0.6037	0.6001	0.6018	0.6086	0.6204	$GD_5\ GD_{11}$
$GC_6\ GC_{12}$	0.6036	0.6080	0.6136	0.6198	0.6259	0.6311	0.6346	0.6359	0.6349	0.6317	$GD_6\ GD_{12}$
	180	170	160	150	140	130	120	110	100	90	θ

$\alpha = 0.6$.

θ	90	100	110	120	130	140	150	160	170	180	
$GA_1\ GA_2$	0.6231	0.6535	0.6980	0.7572	0.8295	0.9070	0.9718	0.9989	0.9746	0.9112	$GF_1\ GF_2$
$GB_1\ GB_6$	0.6922	0.6532	0.6250	0.6075	0.6002	0.6030	0.6160	0.6395	0.6738	0.7184	$GE_1\ GE_6$
$GB_2\ GB_7$	0.6077	0.6002	0.6042	0.6201	0.6485	0.6904	0.7467	0.8157	0.8909	0.9560	$GE_2\ GE_7$
$GB_3\ GB_8$	0.6121	0.6351	0.6710	0.7208	0.7842	0.8569	0.9270	0.9736	0.9762	0.9336	$GE_3\ GE_8$
$GB_4\ GB_9$	0.7084	0.7583	0.8110	0.8564	0.8817	0.8774	0.8452	0.7964	0.7438	0.6958	$GE_4\ GE_9$
$GB_5\ GB_{10}$	0.8246	0.8079	0.7750	0.7347	0.6945	0.6594	0.6318	0.6127	0.6022	0.6003	$GE_5\ GE_{10}$
$GC_1\ GC_7$	0.6139	0.6033	0.5999	0.6038	0.6149	0.6329	0.6572	0.6862	0.7171	0.7455	$GD_1\ GD_7$
$GC_2\ GC_8$	0.6024	0.6006	0.6098	0.6303	0.6627	0.7077	0.7646	0.8294	0.8922	0.9356	$GD_2\ GD_8$
$GC_3\ GC_9$	0.6001	0.6075	0.6269	0.6593	0.7060	0.7672	0.8408	0.9176	0.9780	0.9974	$GD_3\ GD_9$
$GC_4\ GC_{10}$	0.6049	0.6201	0.6459	0.6825	0.7291	0.7830	0.8372	0.8794	0.8958	0.8798	$GD_4\ GD_{10}$
$GC_5\ GC_{11}$	0.6204	0.6363	0.6551	0.6749	0.6929	0.7062	0.7121	0.7093	0.6984	0.6818	$GD_5\ GD_{11}$
$GC_6\ GC_{12}$	0.6317	0.6267	0.6207	0.6144	0.6087	0.6041	0.6010	0.5999	0.6008	0.6036	$GD_6\ GD_{12}$
	90	80	70	60	50	40	30	20	10	0	θ

$\alpha = 0.7$.

θ	0	10	20	30	40	50	60	70	80	90	
$GA_1\ GA_2$	0.9449	0.8919	0.8374	0.7890	0.7502	0.7225	0.7058	0.7000	0.7052	0.7212	$GF_1\ GF_2$
$GB_1\ GB_6$	0.7975	0.8364	0.8740	0.9034	0.9173	0.9118	0.8884	0.8535	0.8143	0.7772	$GE_1\ GE_6$
$GB_2\ GB_7$	0.9712	0.9899	0.9795	0.9438	0.8935	0.8404	0.7923	0.7533	0.7247	0.7070	$GE_2\ GE_7$
$GB_3\ GB_8$	0.9562	0.9114	0.8594	0.8095	0.7671	0.7346	0.7128	0.7016	0.7011	0.7110	$GE_3\ GE_8$
$GB_4\ GB_9$	0.7794	0.7481	0.7239	0.7080	0.7005	0.7017	0.7116	0.7299	0.7561	0.7891	$GE_4\ GE_9$
$GB_5\ GB_{10}$	0.7002	0.7059	0.7187	0.7382	0.7631	0.7915	0.8202	0.8447	0.8603	0.8633	$GE_5\ GE_{10}$
$GC_1\ GC_7$	0.8067	0.8197	0.8241	0.8190	0.8055	0.7863	0.7646	0.7435	0.7252	0.7113	$GD_1\ GD_7$
$GC_2\ GC_8$	0.9509	0.9549	0.9350	0.8973	0.8514	0.8059	0.7663	0.7352	0.7138	0.7022	$GD_2\ GD_8$
$GC_3\ GC_9$	0.9978	0.9790	0.9365	0.8828	0.8291	0.7822	0.7452	0.7191	0.7041	0.7001	$GD_3\ GD_9$
$GC_4\ GC_{10}$	0.9083	0.8818	0.8453	0.8062	0.7699	0.7399	0.7179	0.7045	0.6999	0.7043	$GD_4\ GD_{10}$
$GC_5\ GC_{11}$	0.7592	0.7458	0.7320	0.7194	0.7094	0.7027	0.6999	0.7013	0.7066	0.7155	$GD_5\ GD_{11}$
$GC_6\ GC_{12}$	0.7026	0.7058	0.7097	0.7140	0.7182	0.7217	0.7240	0.7249	0.7242	0.7220	$GD_6\ GD_{12}$
	180	170	160	150	140	130	120	110	100	90	θ

$\alpha = 0.7$.

θ	90	100	110	120	130	140	150	160	170	180	
$GA_1\ GA_2$	0.7212	0.7484	0.7865	0.8345	0.8888	0.9421	0.9830	0.9991	0.9847	0.9449	$GF_1\ GF_2$
$GB_1\ GB_6$	0.7772	0.7457	0.7219	0.7066	0.7002	0.7027	0.7141	0.7342	0.7625	0.7975	$GE_1\ GE_6$
$GB_2\ GB_7$	0.7070	0.7001	0.7039	0.7183	0.7437	0.7797	0.8254	0.8776	0.9298	0.9712	$GE_2\ GE_7$
$GB_3\ GB_8$	0.7110	0.7317	0.7630	0.8044	0.8537	0.9059	0.9521	0.9805	0.9821	0.9562	$GE_3\ GE_8$
$GB_4\ GB_9$	0.7891	0.8260	0.8624	0.8917	0.9073	0.9047	0.8847	0.8527	0.8155	0.7794	$GE_4\ GE_9$
$GB_5\ GB_{10}$	0.8633	0.8529	0.8317	0.8042	0.7752	0.7484	0.7264	0.7106	0.7018	0.7002	$GE_5\ GE_{10}$
$GC_1\ GC_7$	0.7113	0.7028	0.7000	0.7032	0.7121	0.7263	0.7448	0.7661	0.7877	0.8067	$GD_1\ GD_7$
$GC_2\ GC_8$	0.7022	0.7006	0.7089	0.7271	0.7550	0.7920	0.8360	0.8826	0.9241	0.9509	$GD_2\ GD_8$
$GC_3\ GC_9$	0.7001	0.7069	0.7246	0.7534	0.7931	0.8422	0.8967	0.9488	0.9864	0.9978	$GD_3\ GD_9$
$GC_4\ GC_{10}$	0.7043	0.7176	0.7395	0.7694	0.8056	0.8448	0.8813	0.9080	0.9179	0.9083	$GD_4\ GD_{10}$
$GC_5\ GC_{11}$	0.7155	0.7273	0.7408	0.7545	0.7667	0.7754	0.7791	0.7773	0.7703	0.7592	$GD_5\ GG_{11}$
$GC_6\ GC_{12}$	0.7220	0.7187	0.7146	0.7103	0.7062	0.7030	0.7008	0.7000	0.7006	0.7026	$GD_6\ GD_{12}$
	90	80	70	60	50	40	30	20	10	0	θ

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